INHOMOGENEOUS SELF-SIMILAR SETS AND MEASURES

Nina Snigireva

A Thesis Submitted for the Degree of PhD
at the
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I, Nina Snigireva, hereby certify that this thesis, which is approximately 43.000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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Abstract

The thesis consists of four main chapters.
The first chapter includes an introduction to inhomogeneous self-similar sets and measures. In particular, we show that these sets and measures are natural generalizations of the well known self-similar sets and measures. We then investigate the structure of these sets and measures.
In the second chapter we study various fractal dimensions (Hausdorff, packing and box dimensions) of inhomogeneous self-similar sets and compare our results with the well-known results for (ordinary) self-similar sets.
In the third chapter we investigate the L^q spectra and the Renyi dimensions of inhomogeneous self-similar measures and prove that new multifractal phenomena, not exhibited by (ordinary) self-similar measures, appear in the inhomogeneous case. Namely, we show that inhomogeneous self-similar measures may have phase transitions which is in sharp contrast to the behaviour of the L^q spectra of (ordinary) self-similar measures satisfying the Open Set Condition. Then we study the significantly more difficult problem of computing the multifractal spectra of inhomogeneous self-similar measures. We show that the multifractal spectra of inhomogeneous self-similar measures may be non-concave which is again in sharp contrast to the behaviour of the multifractal spectra of (ordinary) self-similar measures satisfying the Open Set Condition. Then we present a number of applications of our results. Many of them are related to the notoriously difficult problem of computing (or simply obtaining non-trivial bounds) for the multifractal spectra of self-similar measures not satisfying the Open Set Condition. More precisely, we will show that our results provide a systematic approach to obtain non-trivial bounds (and in some cases even exact values) for the multifractal spectra of several large and interesting classes of self-similar measures not satisfying the Open Set Condition.
In the fourth chapter we investigate the asymptotic behaviour of the Fourier transforms of inhomogeneous self-similar measures and again we present a number of applications of our results, in particular to non-linear self-similar measures.

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Contents

1 Introduction

Fractal geometry was developed to understand the geometry of irregular sets which cannot be done using methods from classical Euclidean geometry. One of the most famous examples of such irregular sets is the well known Cantor set. The geometry of the Cantor set cannot be easily described using approaches from classical geometry. More precisely, even though the Cantor set is an uncountable set it has Lebesgue measure zero and therefore its size cannot be described by classical measures. However, fractal geometry provides answers to such questions by introducing the notion of fractal measures and dimensions. Namely, fractal measures and dimensions extend the classical concepts of measures and dimensions to non-integer values. Hence, for example, in the case of the Cantor set we can estimate its size by computing its fractal dimension, which will have the value strictly bigger than zero and strictly less than one, indicating that the structure of the Cantor set is more complex than that of a regular geometric set. Irregular sets have been known from the time of Cantor, von Koch, Sierpinski, but only in the 1970s when the term "fractal" was introduced by Mandelbrot in his seminal work [Man77] the study of such irregular sets attracted considerable interest and became widespread. It became apparent that not only such sets provide a better description of natural phenomena but that also the mathematics of such sets is very rich and therefore worth studying.

The concept of self-similarity is one of the central concepts in fractal geometry. Often fractals have some degree of self-similarity. For example, the fore mentioned Cantor set is self-similar, namely it is made of scaled down copies which are geometrically similar to the whole set. It is fairly easy to construct the Cantor set. However, we will now briefly describe the construction of another well know self-similar set, the Sierpinski triangle. The reason for choosing the Sierpinski triangle instead of the Cantor set to illustrate how the concept of self-similarity arises in its construction will become clear later. Namely, we believe that the example of the Sierpinski triangle demonstrates better how the concept of self-similar sets extends to the concept of inhomogeneous self-similar sets. The Sierpinski triangle is constructed from any triangle in a plane by a sequence of deletion operations. For example, take (for simplicity) an equilateral triangle and connect the midpoints of each side to form four separate triangles, and delete the triangle in the center. For each of the three remaining triangles, perform the same procedure and iterate. For a graphical illustration of this construction see Figure 1.0.1. In this work we will consider natural extensions of such self-similar constructions by adding an inhomogeneous set at each stage of the construction. More precisely, take as before an equilateral triangle and connect the midpoints of each side to form four separate triangles, and delete the triangle in the centre but now add an inhomogeneous set in the place of the deleted triangle. Then for each of the three remaining triangles, perform the same procedure by adding the scaled down inhomogeneous sets in the places of deleted triangles and iterate. For a graphical illustration of this inhomogeneous self-similar construction see Figure 1.0.2. The concept of self-similar sets extends readily to self-similar measures. For example, in the case of the Sierpinski triangle we can allocate a mass or a probability to each of the remaining triangles at each stage of the construction. For instance, in the construction we have just described we have chosen each of the remaining three triangles with the same probability, namely a third, in the first stage of the construction, but we can assign different probabilities with the main requirement that the total sum of them is equal to one. This gives rise to self-similar measures. Similarly, we can assign an inhomogeneous self-similar measure to the inhomogeneous self-similar set.
In the early 1980s Hutchinson introduced a general framework for studying self-similar sets and measures [Hut81] and since that time self-similar sets and measures have been studied intensively. inhomogeneous self-similar sets and measures were first considered by Barnsley et al. [BD85, Bar89, Bar] in the late 1980s as a tool for image compression and have subsequently been mentioned in various texts [Bar93, Bar06, BH93, Per94]. However, unlike self-similar sets and measures, inhomogeneous self-similar sets and measures were not that widely studied. One of the main tools to understand the geometry of self-similar sets is to compute its fractal dimensions and there are two main approaches

to study self-similar measures: multifractal analysis and Fourier analysis. Multifractal analysis describes global and local behaviour of a measure of a ball centred at a point x with arbitrary small radius. These global and local behaviours for many “good” measures are related to each other by Multifractal Formalism. This is, for example, the case for self-similar measures satisfying an appropriate separation condition, known as the Open Set Condition. In fact, many results for self-similar measures are obtained under the assumption of the Open Set Condition. Fourier analysis provides the description of the asymptotic behaviour of the Fourier transforms of measures. In this thesis we will continue these lines of investigation for inhomogeneous self-similar sets and measures. To the best of our knowledge very little or nothing has been said about various fractal dimensions of inhomogeneous self-similar sets as well as multifractal analysis and Fourier analysis of inhomogeneous self-similar measures. Moreover, we will also show that our results for multifractal spectra of inhomogeneous self-similar measures provide a systematic approach to study the mutifractal spectra of several large and interesting classes of self-similar measures not satisfying the Open Set Condition.

We will now turn towards the brief description of the main work of this thesis. The work in this thesis is based on the following three research papers joint with L. Olsen: ”L^q spectra and Rényi dimensions of in-homogeneous self-similar measures” [OS07], ”Multifractal spectra of in-homogenous self-similar measures” [OS08b] and ”Inhomogenous self-similar measures and their Fourier transforms”[OS08a].

In the first chapter of the thesis we will discuss in more details inhomogeneous self-similar sets and measures and give precise definitions of these sets and measures. Moreover, we will show that inhomogeneous self-similar sets and measures are natural generalizations of the well known and widely studied self-similar sets and measures. One could have already suspected this from our examples above of the Sierpinski triangle and of the Sierpinski triangle with inhomogeneous set added at each stage of the construction. We will also give a detailed account of the structure of inhomogeneous self-similar sets and measures and draw parallels with the homogeneous and inhomogeneous linear equations.

In the second chapter of the thesis we will study various fractal dimensions of inhomogeneous self-similar sets. Namely, we will give the precise formulae for the Hausdorff and the packing dimensions of inhomogeneous self-similar sets. We will also compute the upper box-counting dimension of these sets under appropriate inhomogeneous separation condition. We also like to stress out that, unlike the upper box-counting dimensions, the Hausdorff and the packing dimensions of inhomogeneous self-similar sets are obtained without assuming any separation conditions.

In the third chapter of the thesis we will first study the L^q spectra and the Rényi dimensions of inhomogeneous self-similar measures. L^q spectra and the Rényi dimensions give the description of the global behavior of a measure of a ball centered at a point x with arbitrary small radius. We will prove that new multifractal phenomena, not exhibited by self-similar measures, appear in the inhomogeneous case. In particular, we show that inhomogeneous self-similar measures may have phase transitions, i.e. points at which the L^q spectra are non-differentiable. This is in sharp contrast to the well known behaviour of the L^q spectra of self-similar measures satisfying the Open Set Condition.
We will then turn towards the study of the significantly more difficult problem of computing the multifratal spectra of inhomogeneous self-similar measures satisfying the appropriate inhomogeneous separation condition, which we will call the inhomogeneous Open Set Condition. Multifractal spectra provide the description of a local behaviour of a measure. More precisely, multifractal spectra of a measure provide the description of a set of points x for which the a measure of a ball centred at a point x with arbitrary small radius behaves like the radius of this ball to some given power. We will again prove that new multifractal phenomena, not exhibited by self-similar measures, appear in the inhomogeneous case. In particular, we will show that the multifractal spectra of inhomogeneous

self-similar measures may be non-concave. This is in sharp contrast to the well known behaviour of the multifractal spectra of self-similar measures satisfying the Open Set Condition. We will then present several applications of our results. We would like to emphasize once more that many of our applications are related to the notoriously difficult problem of computing (or simply obtaining non-trivial bounds) for the multifractal spectra of self-similar measures not satisfying the Open Set Condition. We show that our main results can be applied to obtain non-trivial bounds (and in some cases even exact values) for the multifractal spectra of several large and interesting classes of self-similar measures not satisfying the Open Set Condition.

In the fourth chapter of the thesis we will study the asymptotic behaviour of the Fourier transforms of inhomogeneous self-similar measures. We will then again present a number of applications of our results. In particular, non-linear self-similar measures introduced and investigated by Glickenstein & Strichartz are special cases of inhomogeneous self-similar measures, and as an application of our main results we will obtain simple proofs of generalizations of Glickenstein & Strichartz's results on the asymptotic behaviour of the Fourier transforms of non-linear self-similar measures.

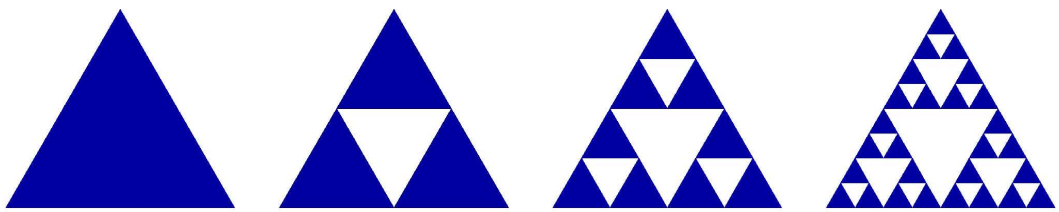

Figure 1.0.1:
First four levels in the construction of the Sierpinski triangle.

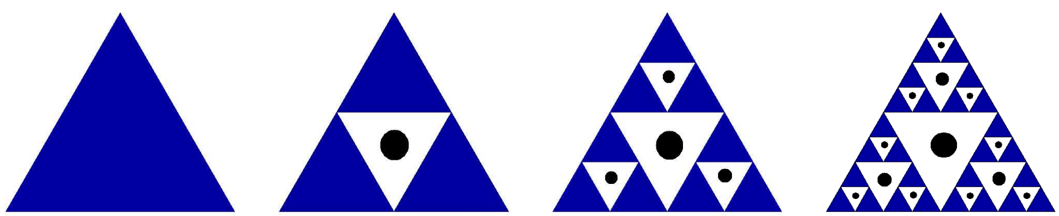

Figure 1.0.2:
First four levels in the construction of the Sierpinski triangle with the inhomogeneous set.

2 Inhomogeneous self-similar sets and measures

2.1 Preliminaries: self-similar sets and measures

The investigation of self-similar sets originates from construction of self-similar sets such as the middle third Cantor set or the Sierpinski triangle. Namely, the sets which are made of scaled down copies that are geometrically similar to the entire set. The first contributions to the theory of self-similarity were made in 1946 by Moran [Mor46]. Moran fractals are constructed in a similar way to the middle third Cantor set with the main differences being that contraction ratios are not required to be constant and the starting set can be of a more general form than the unit interval. In the 1970s the theory of self-similarity became popular due to Mandelbrot who used self-similar sets to analyse various physical phenomena [Man77, Man82]. For example, Cantor sets were used to model noise. For more applications of fractal sets for describing physical phenomena we refer to the books [Man77, Man82, Fal90, Fed88]. In the 1980s Hutchinson introduced the general framework for studying self-similar sets in [Hut81]. We now state the formal definition of self-similar sets. We will denote a self-similar set by $K_\varnothing$. The choice for this notation will be explained in the next section where we will introduce inhomogeneous self-similar sets.

Definition 2.1. (Self-similar sets [Hut81]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities. A compact subset $K_\varnothing$ of $\mathbb{R}^d$ satisfying*

$$K_\varnothing = \bigcup_{i=1}^{N} S_i(K_\varnothing) \tag{2.1}$$

is called a self-similar set associated with the list $(S_1, \ldots, S_N)$.

In the same paper Hutchinson proved that such a set exists and that it is unique. This is the content of the next proposition. The proof of this proposition is based on Banch's fixed point theorem. Since in the next section we will give the proof for existence and uniqueness of inhomogeneous self-similar sets which use similar ideas, we will not state the proof of Proposition 2.2 here.

Proposition 2.2. (Existence and uniqueness of self-similar sets [Hut81]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities. Then there exists a unique non-empty compact subset $K_\varnothing$ of $\mathbb{R}^d$ satisfying (2.1).*

The concept of self-similar sets extends readily to self-similar measures. Namely, introducing self-similar measures supported on self-similar sets provides a better understanding of these sets. This was one of the main motivations in [Hut81] for introducing the general framework for studying self-similar measures. We will now state the formal definition of self-similar measures. Again, the choice for denoting a self-similar measure by μ_0 will become clear in the next section where we will introduce inhomogeneous self-similar measures.

Definition 2.3. (Self-similar measures [Hut81]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities and let $(p_1, \ldots, p_N)$ be a probability vector. A probability measure μ_0 on $\mathbb{R}^d$ such that*

$$\mu_0 = \sum_{i=1}^{N} p_i \mu_0 \circ S_i^{-1} \tag{2.2}$$

is called a self-similar measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N)$.

Proposition 2.4. (Existence and uniqueness of self-similar measures [Hut81]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities and let $(p_1, \ldots, p_N)$ be a probability vector. Then there exists a unique probability measure μ_0 on $\mathbb{R}^d$ satisfying (2.2).*

As we mentioned earlier it is well-known that $\operatorname{supp}\mu_0 = K_\varnothing$.

Self-similar sets and measures have been studied intensively for the past 20 years and there exists a huge body of literature investigating many different aspects of self-similar sets and measures, cf. the textbook [Fal97] and the references therein. In this thesis we investigate various aspects of inhomogeneous self-similar sets and measures.

2.2 Inhomogeneous self-similar sets and measures

We first introduce inhomogeneous self-similar sets and measures and show that these sets and measures are natural generalizations of (ordinary) self-similar sets and measures.
It is natural to view the self-similar equations (2.1) and (2.2) satisfied by $K_\varnothing$ and μ_0 as a homogenous equations. This viewpoint suggests that it would be of interest to investigate the corresponding inhomogeneous equations. This leads to the following definitions.

Definition 2.5. (Inhomogeneous self-similar sets). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities. Also, let C be a compact subset of $\mathbb{R}^d$. A non-empty compact set K_C such that*

$$K_C = \bigcup_{i=1}^{N} S_i(K_C) \cup C . \tag{2.3}$$

is called an inhomogeneous self-similar set associated with the list $(S_1, \ldots, S_N, C)$.

Definition 2.6. (Inhomogeneous self-similar measures). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, N$ be contracting similarities. Also, let $(p_1, \ldots, p_N, p)$ be a probability vector and let ν be a probability measure on $\mathbb{R}^d$ with compact support. A probability measure μ such that*

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + p\nu . \tag{2.4}$$

is called an inhomogeneous self-similar measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N, p, \nu)$.

Observe that an inhomogeneous self-similar set K_C can be viewed as a solution to the inhomogeneous version of equation (2.1) with inhomogeneous term equal to C . Also observe that if $C = \varnothing$, then $K_C = K_\varnothing$; this explains why the self-similar set satisfying (2.1) is denoted by $K_\varnothing$. Similary, an inhomogeneous self-similar measure μ can be viewed as a solution to the inhomogeneous version of equation (2.2) with inhomogeneous term equal to $p\nu$. As it was mentioned in the Introduction inhomogeneous self-similar sets and measures were introduced by Barnsley et al. [BD85, Bar89, Bar] in the 1980s as a tool for image compression and have subsequently been mentioned in various texts [Bar93, Bar06, BH93, Per94]. Barnsley et al. [BD85] also consider a few concrete examples of inhomogeneous self-similar measures. However, their further investigations are restricted to (ordinary) self-similar measures without inhomogeneous terms. In [Bar89, Bar, Bar06] measures μ satisfying (2.4) are called orbital measures and in [BD85, Bar89, Bar93] the inhomogeneous term C in (2.3) and the inhomogeneous term ν in (2.4) are called the condensation set and the condensation measure, respectively. We also note that inhomogeneous equations have been introduced and studied by Jaffard [Jaf97a, Jaf97b] in the context of fractal functions. inhomogeneous self-similar measures may also be viewed as stationary measures of the Markov operator $\mathcal{M}$ introduced in the proof of Proposition 2.8 below. This viewpoint has recently been investigated by Lasota and Myjak and collaborators in a more general setting in a series of papers, cf. for example, [HMS05] and the references therein. Namely, Lasota and Myjak *et al* assume that the probabilities $p_1, \ldots, p_N$ and p depend on $x \in \mathbb{R}^d$, and, in this more general setting, they study conditions guaranteeing the existence of a stationary measure of the corresponding Markov operator $\mathcal{M}$.

2.2.1 Existence and uniqueness of inhomogeneous self-similar sets and measures

Using ideas from [Hut81] it is easily seen that for a given list $(S_1, \ldots, S_N, p_1, \ldots, p_N, p, \nu)$ there exist a unique inhomogeneous self-similar set satisfying (2.3) and measure satisfying (2.4). Indeed, this observation and argument also goes back to Barnsley *et al.* [BD85, Bar, Bar06]; however, for sake of completeness we will sketch the simple proofs here. We also note that the existence of a unique inhomogeneous self-similar set satisfying (2.3) was proved independently from Barnsley *et al.* by Hata in 1986 [Hat86].

Proposition 2.7. (Existence and uniqueness of inhomogeneous self-similar sets. [Bar93, BD85, Hat86], see also [Per94]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities with contractivity factors r_i. Also, let C be a compact subset of $\mathbb{R}^d$. Then there exists a unique non-empty compact subset K_C of $\mathbb{R}^d$ such that*

$$K_C = \bigcup_{i=1}^{N} S_i(K_C) \cup C\,.$$

Proof. Let $\mathtt{K}(\mathbb{R}^d)$ denote the space of all compact non-empty subsets of $\mathbb{R}^d$ and let $r_{\max} = \max_i r_i$. Define the map $\mathcal{T} : \mathtt{K}(\mathbb{R}^d) \to \mathtt{K}(\mathbb{R}^d)$, by

$$\mathcal{T}(A) = \bigcup_{i=1}^{N} S_i(A) \cup C\,,$$

for all A in $\mathtt{K}(\mathbb{R}^d)$. We will show that $\mathcal{T}$ is a contraction with respect to the Hausdorff metric, d_h. It is well known that $(\mathtt{K}(\mathbb{R}^d), d_h)$ is a complete metric space (for the proof see, for example, [Bar93] or classical text books [Eng89, Kec95]). Suppose A, B are in $\mathtt{K}(\mathbb{R}^d)$, then

$$\begin{aligned} d_h(\mathcal{T}(A), \mathcal{T}(B)) &= d_h\left(\bigcup_{i=1}^{N} S_i(A) \cup C, \bigcup_{i=1}^{N} S_i(B) \cup C\right) \\ &\le \max\left(d_h\left(\bigcup_{i=1}^{N} S_i(A), \bigcup_{i=1}^{N} S_i(B)\right), d_h(C, C)\right) \\ &= d_h\left(\bigcup_{i=1}^{N} S_i(A), \bigcup_{i=1}^{N} S_i(B)\right)\,, \end{aligned} \tag{2.5}$$

using

$$d_h(A \cup B, C \cup D) \le \max(d_h(A, C), d_h(B, D)) \tag{2.6}$$

for all A, B, C, D in $\mathtt{K}(\mathbb{R}^d)$ (see, for example, [Bar93]). Hence, applying (2.6) to (2.5) repeatedly, we obtain

$$d_h(\mathcal{T}(A), \mathcal{T}(B)) \le \max_i\,(d_h(S_i(A), S_i(B))) \le [\max_i(r_i)] d_h(A, B) = r_{\max} d_h(A, B)\,.$$

Therefore $\mathcal{T}$ is a contraction map on the complete metric space $(\mathtt{K}(\mathbb{R}^d), d_h)$. Hence, it follows from Banach's fixed-point theorem that $\mathcal{T}$ has a unique fixed point. Namely, there exists a unique compact non-empty set K_C such that $\mathcal{T}(K_C) = K_C$. □

Remark. We also note that from Banach's fixed point theorem

$$K_C = \lim_{n\to\infty} \mathcal{T}^n(E) \tag{2.7}$$

for any E in $\mathtt{K}(\mathbb{R}^d)$, where $\mathcal{T}^n$ is the n^{th} iterate of $\mathcal{T}$. We will use equation (2.7) in the later sections of the thesis.

Proposition 2.8. (Existence and uniqueness of inhomogeneous self-similar measures [BD85, Bar, Bar06]) *Let* $S_i : \mathbb{R}^d \to \mathbb{R}^d$ *for* $i = 1, \ldots, N$ *be contracting similarities with conractivity factors* r_i *. Also, let* $(p_1, \ldots, p_N, p)$ *be a probability vector and let* ν *be a probability measure on* $\mathbb{R}^d$ *with compact support. Then there exists a unique probability measure* μ *such that*

$$\mu = \sum_i p_i \mu \circ S_i^{-1} + p\nu \, .$$

Proof. Let $\mathcal{P}(\mathbb{R}^d)$ be the family of all probability Borel measures on $\mathbb{R}^d$. Define the map $\mathcal{M} : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{M}(\mu) = \sum_i p_i \mu \circ S_i^{-1} + p\nu \, .$$

We will now show that $\mathcal{M}$ is a contraction with respect to the metric,

$$L(\mu_1, \mu_2) = \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} \left| \int f \, d\mu_1 - \int f \, d\mu_2 \right| ,$$

where $\mathrm{Lip}(f)$ is the Lipschitz constant of f . Recall that $r_{\max} = \max_i r_i$. We have,

$$\begin{aligned}
& L(\mathcal{M}(\mu_1), \mathcal{M}(\mu_2)) & (2.8)\\
= \; & \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} \left| \int f \, d\mathcal{M}(\mu_1) - \int f \, d\mathcal{M}(\mu_2) \right| \\
= \; & \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} \left| \left(\sum_i p_i \int (f \circ S_i) \, d\mu_1 + p \int f \, d\nu \right) - \left(\sum_i p_i \int (f \circ S_i) \, d\mu_2 + p \int f \, d\nu \right) \right| \\
= \; & \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} \left| \sum_i p_i \left(\int (f \circ S_i) \, d\mu_1 - \int (f \circ S_i) \, d\mu_2 \right) \right| \\
\leq \; & \sum_i p_i \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} \left| \int (f \circ S_i) \, d\mu_1 - \int (f \circ S_i) \, d\mu_2 \right| \\
= \; & \sum_i p_i \sup_{\substack{f : \mathbb{R}^d \to \mathbb{R}, \\ \mathrm{Lip}(f) \leq 1}} r_i \left| \int r_i^{-1} (f \circ S_i) \, d\mu_1 - \int r_i^{-1} (f \circ S_i) \, d\mu_2 \right| \\
\leq \; & \sum_i p_i r_i \, L(\mu_1, \mu_2) \\
\leq \; & \sum_i p_i r_{\max} \, L(\mu_1, \mu_2) \\
\leq \; & r_{\max} \, L(\mu_1, \mu_2) \, ,
\end{aligned}$$

since $\mathrm{Lip}\left(r_i^{-1} (f \circ S_i)\right) \leq 1$ for all i and $\sum_i p_i \leq 1$.
It is well known that $(\mathcal{P}(\mathbb{R}^d), L)$ is a complete metric space (for a proof see [Hut81]), and it therefore follows immediately from Banach's fixed-point theorem that $\mathcal{M}$ has a unique fixed point, i.e. there exists a unique measure μ such that $\mathcal{M}(\mu) = \mu$. □

2.2.2 Support of inhomogeneous self-similar measures

The support of the inhomogeneous measure μ satisfies the following equation. Namely, if C denotes the support of ν , then the support of μ is equal to the unique non-empty compact set K_C satisfying (2.3). This is the content of the next proposition.

Proposition 2.9. *Let μ be an inhomogeneous self-similar measure satisfying (2.4) and let C be the support of ν. Then the support of μ is equal to the unique non-empty compact set K_C satisfying (2.3).*

Proof. First we prove that $\operatorname{supp}\mu \subseteq \cup_i S_i(\operatorname{supp}\mu) \cup C$. Indeed, applying equation (2.4) to $\cup_i S_i(\operatorname{supp}\mu) \cup C$, we obtain

$$\begin{aligned}
\mu\Big(\bigcup_i S_i(\operatorname{supp}\mu) \cup C\Big) &= \sum_k p_k \mu\Big(S_k^{-1}\Big(\bigcup_i S_i(\operatorname{supp}\mu) \cup C\Big)\Big) + p\nu\Big(\bigcup_i S_i(\operatorname{supp}\mu) \cup C\Big) \\
&= \sum_k p_k \mu\Big(S_k^{-1}\Big(\bigcup_i S_i(\operatorname{supp}\mu) \cup C\Big)\Big) + p \\
&\geq \sum_k p_k \mu\Big(S_k^{-1} S_k(\operatorname{supp}\mu) \cup S_k^{-1} C\Big) + p \\
&= \sum_k p_k + p \\
&= 1\,.
\end{aligned}$$

Thus, since μ is a probability measure we conclude that $\mu\Big(\cup_i S_i(\operatorname{supp}\mu) \cup C\Big) = 1$. Hence $\operatorname{supp}\mu \subseteq \cup_i S_i(\operatorname{supp}\mu) \cup C$.
Next, we prove that $\cup_i S_i(\operatorname{supp}\mu) \cup C \subseteq \operatorname{supp}\mu$. Noting that $C \subseteq \operatorname{supp}\mu$ and applying equation (2.4) to $\operatorname{supp}\mu$, we obtain

$$\begin{aligned}
1 &= \mu(\operatorname{supp}\mu) \\
&= \sum_i p_i \mu\big(S_i^{-1}(\operatorname{supp}\mu)\big) + p\nu(\operatorname{supp}\mu) \\
&= \sum_i p_i \mu\big(S_i^{-1}(\operatorname{supp}\mu)\big) + p \\
&\leq \sum_i p_i + p \\
&= 1\,,
\end{aligned}$$

whence $\sum_i p_i \mu\big(S_i^{-1}(\operatorname{supp}\mu)\big) + p = 1$. Since also $\sum_i p_i + p = 1$, we conclude from this that $\mu(S_i^{-1}(\operatorname{supp}\mu)) = 1$ for all i. Hence $\operatorname{supp}\mu \subseteq S_i^{-1}(\operatorname{supp}\mu)$, implying that $S_i(\operatorname{supp}\mu) \subseteq \operatorname{supp}\mu$ for all i. Thus $\cup_i S_i(\operatorname{supp}\mu) \subseteq \operatorname{supp}\mu$ and therefore $\cup_i S_i(\operatorname{supp}\mu) \cup C \subseteq \operatorname{supp}\mu$. Hence $\operatorname{supp}\mu = \cup_i S_i(\operatorname{supp}\mu) \cup C$. Therefore $\operatorname{supp}\mu$ is the unique non-empty compact set satisfying (2.3). $\square$

2.3 The structure of inhomogeneous self-similar sets and measures.

In this section we investigate the structure of inhomogeneous self-similar sets and measures. We begin by introducing some notation. For a non-negative integer n, let

$$\begin{aligned}
\Sigma^n &= \{1, \dots, N\}^n\,, \\
\Sigma^* &= \bigcup_n \Sigma^n\,,
\end{aligned}$$

i.e. Σ^n is the family of all finite strings $\mathbf{i} = i_1 \dots i_n$ of length n with entries $i_j \in \{1, \dots, N\}$, and Σ^* denotes the family of all finite strings $\mathbf{i} = i_1 \dots i_n$ with entries $i_j \in \{1, \dots, N\}$. For $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$, we will write $|\mathbf{i}| = n$ for the length of $\mathbf{i}$, and if m is an integer with $m \leq n$, we will write $\mathbf{i}|m = i_1 \dots i_m$ for the truncation of $\mathbf{i}$ to the m'th place. Finally, for $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$,

we write $S_{\mathbf{i}} = S_{i_1} \circ \cdots \circ S_{i_n}$ and $p_{\mathbf{i}} = p_{i_1} \cdots p_{i_n}$ and $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$.
Next, let

$$\mathsf{B}(\mathbb{R}^d) = \left\{ B \subseteq \mathbb{R}^d \,\middle|\, B \text{ is bounded} \right\},$$

and define $\mathcal{S} : \mathsf{B}(\mathbb{R}^d) \to \mathsf{B}(\mathbb{R}^d)$ by

$$\mathcal{S}(B) = \bigcup_i S_i(B).$$

Using this definition of $\mathcal{S}$, it follows that $K_\varnothing$ is the unique non-empty compact set such that $K_\varnothing = \mathcal{S}(K_\varnothing)$. Similarly, it follows that K_C is the unique non-empty compact set such that $K_C = \mathcal{S}(K) \cup C$. The next theorem provides detailed information about the structure of the solutions $X \in \mathsf{B}(\mathbb{R}^d)$ to the homogenous equation $X = \mathcal{S}(X)$ and to the inhomogeneous equation $X = \mathcal{S}(X) \cup C$. The theorem also shows that there is a close connection between the sets K_C and $K_\varnothing$.

Theorem 2.10. *Let*

$$O = \bigcup_{\mathbf{i} \in \mathbf{\Sigma}^*} S_{\mathbf{i}}(C). \tag{2.9}$$

1. *(i) The set $K_\varnothing \in \mathsf{B}(\mathbb{R}^d)$ satisfies the homogenous equation $K_\varnothing = \mathcal{S}(K_\varnothing)$.*
   (ii) If a set $X \in \mathsf{B}(\mathbb{R}^d)$ satisfies the homogenous equation $X = \mathcal{S}(X)$, then $X \subseteq \overline{X} = K_\varnothing$.

   In particular, this shows that $K_\varnothing$ is the biggest set X in $\mathsf{B}(\mathbb{R}^d)$ satisfying the homogenous equation $X = \mathcal{S}(X)$.

2. *(i) The set $O \in \mathsf{B}(\mathbb{R}^d)$ satisfies the inhomogeneous equation $O = \mathcal{S}(O) \cup C$.*
   (ii) The set $K_C \in \mathsf{B}(\mathbb{R}^d)$ satisfies the inhomogeneous equation $K_C = \mathcal{S}(K_C) \cup C$.
   (iii) If a set $X \in \mathsf{B}(\mathbb{R}^d)$ satisfies the inhomogeneous equation $X = \mathcal{S}(X) \cup C$, then $O \subseteq X \subseteq \overline{X} = K_C$.

   In particular, this shows that O is the smallest set X in $\mathsf{B}(\mathbb{R}^d)$ satisfying the inhomogeneous equation $X = \mathcal{S}(X) \cup C$, and that K_C is the biggest set X in $\mathsf{B}(\mathbb{R}^d)$ satisfying the inhomogeneous equation $X = \mathcal{S}(X) \cup C$.

3. *If a set $X \in \mathsf{B}(\mathbb{R}^d)$ satisfies the inhomogeneous equation $X = \mathcal{S}(X) \cup C$, then*

$$K_C = K_\varnothing \cup X.$$

The proof of Theorem 2.10 is given in Section 2.3.1.

Remark. In [Bar06] the set O is called the orbital set (we will provide an explanation of this terminology in the remark following Theorem 2.11). We also note that Part 2.(i) of Theorem 2.10 (saying that $O = \mathcal{S}(O) \cup C$) is proved in [Bar06].

Remark. We see from Theorem 2.10 that $K_\varnothing$ is the biggest set X in $\mathsf{B}(\mathbb{R}^d)$ satisfying the homogenous equation $X = \mathcal{S}(X)$, and that K_C is the biggest set X in $\mathsf{B}(\mathbb{R}^d)$ satisfying the inhomogeneous equation $X = \mathcal{S}(X) \cup C$. We also see that if $X \in \mathsf{B}(\mathbb{R}^d)$ is an arbitrary solution to the inhomogeneous equation $X = \mathcal{S}(X) \cup C$, then

$$K_C = K_\varnothing \cup X,$$

i.e.

$$\begin{aligned}
&\text{" the biggest solution } X \text{ to the inhomogeneous equation } X = \mathcal{S}(X) \cup C\text{"}\\
&= \text{" the biggest solution } X \text{ to the homogenous equation } X = \mathcal{S}(X)\text{"} \qquad (2.10)\\
&\quad \cup \text{" an arbitrary solution } X \text{ to the inhomogeneous equation } X = \mathcal{S}(X) \cup C\text{"} .
\end{aligned}$$

This result is clearly reminiscent of the structure of the set of solutions to inhomogeneous linear equations. We will now explain this in more detail. Fix a vector space V. Let $A : V \to V$ be linear and let $c \in V$. Write Λ_0 for the complete solution to the homogenous equation $x = Ax$, i.e.

$$\Lambda_0 = \left\{ x \in V \,\middle|\, x = Ax \right\},$$

and write Λ_c for the complete solution to the inhomogeneous equation $x = Ax + c$, i.e.

$$\Lambda_c = \left\{ x \in V \,\middle|\, x = Ax + c \right\}.$$

It is clear that if $x \in V$ satisfies the inhomogeneous equation $x = Ax + c$, then

$$\Lambda_c = \Lambda_0 + x ,$$

i.e.

$$\begin{aligned}
&\text{" the complete solution to the inhomogeneous equation } x = Ax + c\text{"}\\
&= \text{" the complete solution to the homogenous equation } x = Ax\text{"} \qquad (2.11)\\
&\quad + \text{" an arbitrary solution } x \text{ to the inhomogeneous equation } x = Ax + c\text{"} .
\end{aligned}$$

The reader will notice the similarity between the statements in 2.10 and 2.11

We now consider some further consequences of Theorem 2.10 .
The first result shows that inhomogeneous self-similar sets and measures can be represented as limits involving only the inhomogeneous terms C and ν. This follows easily from Theorem 2.10 and is the content of Theorem 2.11. Before we state Theorem 2.11 we recall the following notation. Namely, we denote the family of non-empty compact subsets of $\mathbb{R}^d$ by $\mathtt{K}(\mathbb{R}^d)$, and we equip $\mathtt{K}(\mathbb{R}^d)$ with the Hausdorff metric d_h. Also, we denote the family of Borel probability measures on $\mathbb{R}^d$ by $\mathcal{P}(\mathbb{R}^d)$, and we equip $\mathcal{P}(\mathbb{R}^d)$ with the weak topology w.

Theorem 2.11.

1. *We have* $K_C = \lim_{n\to\infty} \bigcup_{\mathbf{i}\in\Sigma^*, |\mathbf{i}|\le n} S_{\mathbf{i}}(C)$ *where the convergence is in* $\big(\mathtt{K}(\mathbb{R}^d), d_h\big)$.

2. *We have* $\mu = \lim_{n\to\infty} \sum_{\mathbf{i}\in\Sigma^*, |\mathbf{i}|\le n} \frac{p_{\mathbf{i}}}{\sum_{\mathbf{j}\in\Sigma^*,\, |\mathbf{j}|\le n} p_{\mathbf{j}}}\, \nu \circ S_{\mathbf{i}}^{-1}$ *where the convergence is in* $\big(\mathcal{P}(\mathbb{R}^d), w\big)$.

Proof. 1. Write $C_n = \bigcup_{\mathbf{i}\in\Sigma^*,\, |\mathbf{i}|\le n} S_{\mathbf{i}}C$. Then $C_1 \subseteq C_2 \subseteq C_3 \subseteq \dots$ and $\bigcup_n C_n = O$ (where O denotes the orbital set in (2.9)). We conclude from this that $(C_n)_n$ is convergent in $\big(\mathtt{K}(\mathbb{R}^d), d_h\big)$ with $\lim_n C_n = \overline{\bigcup_n C_n} = \overline{O}$. However, Theorem 2.10 shows that $\overline{O} = K_C$, whence $\lim_n C_n = \overline{O} = K_C$.
2. Define probability measures ν_n and μ_n by

$$\begin{aligned}\nu_n &= \sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|<n} \frac{p_{\mathbf{i}}}{\sum_{\mathbf{j}\in\Sigma^*,\,|\mathbf{j}|<n} p_{\mathbf{j}}}\,\nu\circ S_{\mathbf{i}}^{-1} = \frac{p}{1-(1-p)^n}\sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|<n} p_{\mathbf{i}}\,\nu\circ S_{\mathbf{i}}^{-1}\,, \\ \mu_n &= \sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|=n} \frac{p_{\mathbf{i}}}{\sum_{\mathbf{j}\in\Sigma^*,\,|\mathbf{j}|=n} p_{\mathbf{j}}}\,\mu\circ S_{\mathbf{i}}^{-1} = \frac{1}{(1-p)^n}\sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|=n} p_{\mathbf{i}}\,\mu\circ S_{\mathbf{i}}^{-1}\,.\end{aligned} \tag{2.12}$$

Iterating (2.4) shows that

$$\mu = \sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|=n} p_{\mathbf{i}}\mu\circ S_{\mathbf{i}}^{-1} + p\sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|<n} p_{\mathbf{i}}\nu\circ S_{\mathbf{i}}^{-1} = (1-p)^n\mu_n + (1-(1-p)^n)\nu_n$$

for all positive integers n. It follows immediately from this that $\nu_n\to\mu$ in $\big(\mathcal{P}(\mathbb{R}^d),w\big)$. □

Remark. The reader will notice the similarity between the expressions for K_C and μ in the previous theorem.

Remark. The non-trivial Part 2 of Theorem 2.11 not relying on Theorem 2.10 also appears in [Bar06]. However, we have decided to included both the statement of Part 2 and the simple proof for completeness.

Remark. We can now provide an explanation of why the set O and the measure μ are called the orbital set and the orbital measure in [Bar, Bar06]. Let $\tilde{s}=\{S_{\mathbf{i}}\,|\,\mathbf{i}\in\Sigma^*\}$ denote the semigroup of mappings from $\mathbb{R}^d\to\mathbb{R}^d$ generated by the S_i 's and the identity map, and write $\mathcal{O}(C)=\{S_{\mathbf{i}}C\,|\,\mathbf{i}\in\Sigma^*\}$ for the $\tilde{s}$-orbit of C and write $\mathcal{O}(\nu)=\{\nu\circ S_{\mathbf{i}}^{-1}\,|\,\mathbf{i}\in\Sigma^*\}$ for the $\tilde{s}$-orbit of ν. Using this notation the set $O=\cup_{\mathbf{i}\in\Sigma^*}S_{\mathbf{i}}C$ is simply the union of the sets in the $\tilde{s}$-orbit of C, and it follows from Theorem 2.11 (by letting n tend to ∞ in (2.12)) that the measure $\mu=p\sum_{\mathbf{i}\in\Sigma^*}p_{\mathbf{i}}\;\nu\circ S_{\mathbf{i}}^{-1}$ is simply a suitably weighted sum of the measures in the $\tilde{s}$-orbit of ν. This explains why K_C is called the orbital set and why μ is called the orbital measure.

In Theorem 2.12 below we present a further surprising consequence of Theorem 2.10 . Namely, it may happen that the set $K_\varnothing$ has zero μ measure. This is in sharp contrast to the behaviour of (ordinary) self-similar sets and measures. Indeed, if μ_0 denotes the self-similar measure satisfying (2.2), then $K_\varnothing$ has full μ_0 measure, i.e. $\mu_0(K_\varnothing)=1$. However, at this stage we would like emphasize that even though $K_\varnothing$ can have zero μ measure, the multifractal structure of μ which we will discuss in Section 4.3 of the thesis is non-trivial and, in general, significantly different from the multifractal spectra of ν.

Theorem 2.12.

1. *Assume that the sets $(S_{\mathbf{i}}C)_{\mathbf{i}\in\Sigma^*}$ are pairwise disjoint and that $p\neq 0$. Then the orbital set O (see (2.9)) has full μ measure, i.e. $\mu(O)=1$.*
2. *Assume that the sets $(S_{\mathbf{i}}C)_{\mathbf{i}\in\Sigma^*}$ are pairwise disjoint and disjoint from $K_\varnothing$ and that $p\neq 0$. Then the orbital set O (2.9) has full μ measure and $K_\varnothing$ has zero μ measure, i.e. $\mu(O)=1$ and $\mu(K_\varnothing)=0$.*

Proof. 1. It follows from the definition of μ that $\mu(C)=\sum_i p_i\mu(S_i^{-1}C)+p\nu(C)\geq p\nu(C)=p$, and iterating (2.4) therefore shows that $\mu(S_{\mathbf{i}}C)=\sum_{\mathbf{j}\in\Sigma^*,\,|\mathbf{j}|=n}p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}S_{\mathbf{i}}C)+p\sum_{\mathbf{j}\in\Sigma^*,\,|\mathbf{j}|<n}p_{\mathbf{j}}\nu(S_{\mathbf{j}}^{-1}S_{\mathbf{i}}C)$ $\geq p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}S_{\mathbf{i}}C)=p_{\mathbf{i}}\mu(C)\geq p_{\mathbf{i}}p$, for all $\mathbf{i}\in\Sigma^*$. We conclude from this and the fact that the sets $(S_{\mathbf{i}}C)_{\mathbf{i}\in\Sigma^*}$ are pairwise disjoint that $1\geq\mu(O)=\sum_{\mathbf{i}\in\Sigma^*}\mu(S_{\mathbf{i}}C)\geq\sum_{\mathbf{i}\in\Sigma^*}p_{\mathbf{i}}p=p\sum_{n\geq 0}\sum_{\mathbf{i}\in\Sigma^*,|\mathbf{i}|=n}p_{\mathbf{i}}$ $=p\sum_{n\geq 0}(1-p)^n=1$.
2. Since the sets $(S_{\mathbf{i}}C)_{\mathbf{i}\in\Sigma^*}$ are pairwise disjoint and disjoint from $K_\varnothing$, we deduce that O and $K_\varnothing$ are disjoint. However, as $\mu(O)=1$ (by Part 1) and $K_C=O\cup K_\varnothing$ (by Theorem 2.12), this implies that $1=\mu(K_C)=\mu(O)+\mu(K_\varnothing)=1+\mu(K_\varnothing)$, whence $\mu(K_\varnothing)=0$. □

The reader is referred to Example 4.40 in Section 4.3.2 for an example of a construction for which the sets $(S_{\mathbf{i}}C)_{\mathbf{i}\in\Sigma^*}$ are pairwise disjoint and disjoint from $K_\varnothing$.

2.3.1 Proof of Theorem 2.10

The purpose of this section is to prove Theorem 2.10. To prove this theorem we would need to use Lemma 3.9.

Proof of Theorem 2.10
1. It follows from the definition of $K_\varnothing$ that $K_\varnothing = \mathcal{S}(K_\varnothing)$. Next, if $X \in \mathsf{B}(\mathbb{R}^d)$ satisfies $X = \mathcal{S}(X)$, then it is easily seen that $\overline{X} = \overline{\mathcal{S}(X)} = \mathcal{S}(\overline{X})$. However, since $K_\varnothing$ is the unique non-empty compact set with $K_\varnothing = \mathcal{S}(K_\varnothing)$, we now conclude that $\overline{X} = K_\varnothing$.
2. It follows from the definition of K_C that $K_C = \mathcal{S}(K_C) \cup C$. It also follows easily from the definition of O that $O = \mathcal{S}(O) \cup C$. Next, if $X \in \mathsf{B}(\mathbb{R}^d)$ satisfies $X = \mathcal{S}(X) \cup C$, then it is easily seen that $\overline{X} = \overline{\mathcal{S}(X) \cup C} = \mathcal{S}(\overline{X}) \cup C$. However, since K_C is the unique non-empty compact set with $K_C = \mathcal{S}(K_C) \cup C$, we now conclude that $\overline{X} = K_C$. Finally, we prove that $O \subseteq X$. To prove this note that it follows easily by iterating the equation $X = \mathcal{S}(X) \cup C$ that

$$X = \bigcup_{\mathbf{i}\in\Sigma^*, |\mathbf{i}|=n} S_{\mathbf{i}}(X) \cup \bigcup_{\mathbf{i}\in\Sigma^*, |\mathbf{i}|<n} S_{\mathbf{i}}(C) \supseteq \bigcup_{\mathbf{i}\in\Sigma^*, |\mathbf{i}|<n} S_{\mathbf{i}}(C)$$

for all positive integers n . Taking union over all positive integers now gives $X \supseteq \cup_n \cup_{|\mathbf{i}|<n} S_{\mathbf{i}}(C) = O$.
3. We will show that $K_C = K_\varnothing \cup X$. Indeed, using Lemma 3.9 and Part 2 we conclude that

$$\begin{aligned} K_C &= K_\varnothing \cup O && \text{[by Lemma 3.9]} \\ &\subseteq K_\varnothing \cup X && \text{[since } O \subseteq X \text{ by Part 2]} \\ &\subseteq K_\varnothing \cup K_C && \text{[since } X \subseteq K_C \text{ by Part 2]} \\ &= K_C\,. && \text{[by Lemma 3.9]} \end{aligned}$$

This completes the proof.

3 Dimensions of inhomogeneous self-similar sets

3.1 Preliminaries: fractal measures and dimensions

In this section we will give an overview of the most commonly used fractal measures and dimensions. Such measures and dimensions are important for understanding the geometry of fractal sets, in particular the geomery of self-similar sets and their natural generalizations inhomogeneous self-similar sets discussed in the previous section. The idea behind most definitions of dimension is to measure a set at a particular scale in such a way that irregularities that occur at the scale less than the one we are measuring at are ignored and see how these measurements behave as we decrease the size of the scale, cf. the textbook [Fal90]. We will first introduce the box-counting dimension since the definition of box-counting dimension is conceptually the easiest. Namely, unlike Hausdorff and packing dimensions box-counting dimension is not defined in terms of measures.

3.1.1 Box dimensions

Box dimension was first defined in the late 1920s. It became one of the most widely used fractal dimensions. Firstly, it is relatively easy to calculate the box dimension for concrete cases. For example, it is easy to compute that the box dimension of the middle third Cantor set equals to $\frac{\log 2}{\log 3}$. Secondly, the box dimension is often used for numerical and experimental purposes in sciences. There are many equivalent ways to define the box dimension. For instance, let E be a non-empty bounded subset of $\mathbb{R}^d$ and let $N_\delta(E)$ denote the smallest number of sets of diameter at most δ (for $\delta > 0$) that cover E . Then it is natural to expect that $N_\delta(E)$ might be proportional to some power of $\frac{1}{\delta}$, namely we may expect that we can find some positive number s such that

$$N_\delta(E) \sim \delta^{-s} \quad \text{for } \delta \text{ close to } 0.$$

This leads to the following formal definition of the box dimension.

Definition 3.1. Box dimension. *The lower and upper box dimensions of a subset* E *of* $\mathbb{R}^d$ *are defined by*

$$\underline{\dim}_{\mathrm{B}}(E) = \liminf_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta}$$

and

$$\overline{\dim}_{\mathrm{B}}(E) = \limsup_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta}$$

where $N_\delta(E)$ *is the smallest number of sets of diameter at most* δ *that cover* E *. Alternatively,* $N_\delta(E)$ *can denote either the smallest number of closed balls of radius* δ *that cover* E *or the largest number of disjoint balls of radius* δ *with centres in* E *.*
If $\underline{\dim}_{\mathrm{B}}(E)$ *and* $\overline{\dim}_{\mathrm{B}}(E)$ *coincide then the common value is called the box dimension of* E *and is denoted by* $\dim_{\mathrm{B}}(E)$ *.*

Remark. On the one hand the concept that $\underline{\dim}_{\mathrm{B}}(E)$ can be defined using economical coverings by small balls of equal radius relates to the concept of the Hausdorff dimension. On the other hand the concept that $\overline{\dim}_{\mathrm{B}}(E)$ can be defined using efficient packings by disjoint balls of equal radius that are as dense as possible will form the basis for the definition of the packing dimension in the later section. See [Fal90] for more details. This shows that the definitions of the Hausdorff and the packing dimensions are dual to each other.

In the later section we will need the following equivalent definition of the box dimension.

Definition 3.2. *The lower and upper box dimensions of a subset E of $\mathbb{R}^d$ are defined by*

$$\underline{\dim}_{\mathrm{B}}(E) = d - \limsup_{\delta \searrow 0} \frac{\log \mathcal{L}^d(B(E,\delta))}{\log \delta}$$

and

$$\overline{\dim}_{\mathrm{B}}(E) = d - \liminf_{\delta \searrow 0} \frac{\log \mathcal{L}^d(B(E,\delta))}{\log \delta}\,,$$

where $B(E,\delta) = \left\{ x \in \mathbb{R}^d \,\middle|\, \operatorname{dist}(x,E) \le \delta \right\}$ *and* $\mathcal{L}^d$ *denotes the* d*-dimensional Lebesgue measure.*

Box-counting dimension satisfies certain basic properties which we will list at the end of the section since we want to compare and contrast these properties with the properties of the Hausdorff and packing dimensions. However, we want to mention one major disadvantage of the box dimension, namely

$$\underline{\dim}_{\mathrm{B}}(E) = \underline{\dim}_{\mathrm{B}}(\overline{E}) \text{ and } \overline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(\overline{E})\,, \tag{3.1}$$

where $\overline{E}$ denotes the closure of E. In particular, it follows from (3.1) that the countable set of rational numbers in the interval [0,1] has box dimension equal to 1. This implies that ("small") countable sets can have non-zero box dimension, reducing the usefulness of box dimension. We also note that due to this property box dimension is not used in computing multifractal spectra discussed in Section 4.1 As we will see in the next sections, this disadvantages are not manifested in Hausdorff and packing dimensions whose definitions are based on measures.

3.1.2 Hausdorff measure and dimension

Hausdorff measure is the generalisation of the Carathéodory measure introduced by Constantin Carathéodory in 1914 as a tool for measuring the s-dimensional volume of a set, where s is a non-negative integer. In 1919 Felix Hausdorff extended Carathéodory's notion of s-dimensional volume of a set to non-integer values of s. Namely, Hausdorff introduced the following measure.

Definition 3.3. Hausdorff measure. *For a positive real number $s \ge 0$, the s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ of a set E is defined by*

$$\mathcal{H}^s(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E)\,,$$

where $\mathcal{H}^s_\delta(E)$ is the δ approximative s-dimensional Hausdorff measure of a set E defined by

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) < \delta \right\}.$$

From the definition of the s-dimensional Hausdorff measure it is easily seen that there exists a unique number $\dim_{\mathrm{H}}(E)$ such that

$$\mathcal{H}^s(E) = \begin{cases} 0 & \text{if } \dim_{\mathrm{H}}(E) < s, \\ \infty & \text{if } \dim_{\mathrm{H}}(E) > s. \end{cases}$$

This leads to the following formal definition of the Hausdorff dimension.

Definition 3.4. Hausdorff dimension. *The Hausdorff dimension $\dim_{\mathrm{H}}(E)$ of a set E is defined by*

$$\dim_{\mathrm{H}}(E) = \inf\{s \mid \mathcal{H}^s(E) = 0\} = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$$

We now state and compare the main properties of the Hausdorff and box dimensions. We note that we will state only those main properties which we will use in the later sections of the thesis. First we will list the properties which hold for both box (upper and lower) and Hausdorff dimensions, namely:

- If $A_1 \subseteq A_2$ then $\dim A_1 \leq \dim A_2$.
- If A is a finite set then $\dim A = 0$.
- If S is a similarity transformation then $\dim S(A) = \dim A$.

Hausdorff and upper box dimensions are finitely stable, namely $\dim \cup_{i=1}^{n} A_i = \max_{1 \leq i \leq n} \dim A_i$. Hausdorff dimension is countably stable, namely $\dim \cup_{i=1}^{\infty} A_i = \sup_{1 \leq i < \infty} \dim A_i$.

3.1.3 Packing measure and dimension

As it was noted earlier (see the remark following Definition 3.1) the packing measure is dual to the Hausdorff measure, namely the packing measure is defined using efficient packings which is dual to the definition of the Hausdorff measure by considering economical coverings. The packing measure like the Hausdorff measure gives rise to a dimension. The packing measure and the packing dimension were introduced by Tricot [Tri82] in 1982. Even though the definition of the packing measure is much more recent, nowadays the packing measure is considered as important as Hausdorff measure. Indeed, many Hausdorff measure properties have dual packing measure properties, and it is widely believed that an understanding of both the Hausdorff dimension and the packing dimension of a fractal set provides the basis for a substantially better understanding of the underlying geometry of the set. We will now define the packing measure and dimension. Let $E \subseteq \mathbb{R}^d$ and $\delta > 0$. First, recall that a countable family $(B(x_i, r_i))_i$ of closed balls in $\mathbb{R}^d$ is called a centred δ -packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ for all $i \neq j$.

Definition 3.5. Packing pre-measure. *For a positive real number $s \geq 0$, the s -dimensional packing pre-measure $\overline{\mathcal{P}}^s(E)$ of a set E is defined by*

$$\overline{\mathcal{P}}^s(E) = \inf_{\delta > 0} \overline{\mathcal{P}}^s_\delta(E) ,$$

where $\overline{\mathcal{P}}^s_\delta(E)$ is defined by

$$\overline{\mathcal{P}}^s_\delta(E) = \sup \left\{ \sum_{i=1}^{\infty} (2r_i)^s \;\middle|\; (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\} .$$

Unfortunately, $\overline{\mathcal{P}}^s$ is not necessarily countably subadditive and therefore not necessarily a measure. However, we can modify the definition of $\overline{\mathcal{P}}^s$ to obtain the s -dimensional packing measure $\mathcal{P}^s(E)$ of E as follows.

Definition 3.6. Packing measure. *For a positive real number $s \geq 0$, the s -dimensional packing measure $\mathcal{P}^s(E)$ of a set E is defined by*

$$\mathcal{P}^s(E) = \inf_{E \subseteq \cup_{i=1}^{\infty} E_i} \sum_i \overline{\mathcal{P}}^s(E_i) .$$

We now can define the packing dimension analogously to the Hausdorff dimension.

Definition 3.7. Packing dimension. *The packing dimension $\dim_{\mathrm{P}}(E)$ of a set E is defined by*

$$\dim_{\mathrm{P}}(E) = \inf \{ s \,|\, \mathcal{P}^s(E) = 0 \} = \sup \{ s \,|\, \mathcal{P}^s(E) = \infty \}$$

The packing dimension satisfies the same properties as the Hausdorff dimension (see above).

3.1.4 Hausdorff, Packing and Box dimensions of self-similar sets.

The fractal dimensions of (ordinary) self-similar sets satisfying (2.1) have been studied by Moran in [Mor46] and later by Hutchinson in [Hut81]. Nowadays it is well known what Hausdorff, packing and box dimensions of (ordinary) self-similar sets satisfying the Open Set Condition (OSC) are. Recall that the OSC says that there exists an open, non-empty and bounded subset U of $\mathbb{R}^d$ with $\cup_i S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \varnothing$ for all $i \neq j$. We will now state these well known results saying that Hausdorff, packing and box dimensions of (ordinary) self-similar sets satisfying the OSC coincide. For the proof of these results see, for example, [Hut81, Fal90].

Theorem 3.8. (See [Mor46, Hut81]). *Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities and let r_i denote the contracting ratio of S_i. Also, let $K_\varnothing$ be the (homogeneous) self-similar set satisfying (2.1). Finally, let s be the unique non-negative solution of*

$$\sum_i r_i^s = 1 .$$

If the Open Set Condition is satisfied, then

$$\dim_{\mathrm{H}} K_\varnothing = \dim_{\mathrm{P}} K_\varnothing = \dim_{\mathrm{B}} K_\varnothing = s . \tag{3.2}$$

3.2 Dimensions of inhomogeneous self-similar sets

It was outlined in the section 3.1 that the study of dimensions of fractal sets is important for understanding the geometry of these sets. The main purpose of this section is to investigate various fractal dimensions of inhomogeneous self-similar sets and compare our results with the result for (ordinary) self-similar sets. Before we state our main results we will need the following important property of the inhomogeneous self-similar set K_C satisfying (2.3). Namely, we want to relate the set K_C to the (ordinary) self-similar set $K_\varnothing$ satisfying (2.1). This leads to the following lemma.

Lemma 3.9. *We have*

$$K_C = K_\varnothing \cup O . \tag{3.3}$$

Proof.
We will prove this lemma in two ways.
1. Geometric approach.
Observe that $O \subseteq K_C$ and $K_\varnothing \subseteq K_C$. Hence $O \cup K_\varnothing \subseteq K_C$.
Next, we prove that $K_C \subseteq O \cup K_\varnothing$.
It suffices to show that $(\overline{O} \setminus O) \subseteq K_\varnothing$, since $K_C = \overline{O} = (\overline{O} \setminus O) \cup O$.
Assume that $x \in (\overline{O} \setminus O)$ and recall that $O = C \cup \bigcup_i S_i(O)$.
Hence, we have

$$\begin{aligned} x \in \overline{O} &= \overline{C \cup \bigcup_i S_i(O)} \\ &= C \cup \overline{\bigcup_i S_i(O)} . \end{aligned}$$

Thus, $x \in \overline{\bigcup_i S_i(O)}$ since x is not in O and in particular x is not in C.
Therefore,

$$x \in \overline{\bigcup_i S_i(O)} = \overline{\bigcup_i S_i\left(C \cup \bigcup_j S_j(O)\right)}$$

$$= \overline{\bigcup_{i,j} S_{i,j}(O)} \cup \left(\bigcup_{i} S_i(C) \right) .$$

Thus, $x \in \overline{\bigcup_{i,j} S_{i,j}(O)}$ since x is not in O and in particular x is not in $\bigcup_i S_i(C)$. Repeating this process, we obtain that

$$x \in \overline{\bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(O)} \quad \text{for all } n .$$

Let X be a compact subset of $\mathbb{R}^d$ such that $O \subseteq X$ and $K_\varnothing \subseteq X$. Recall that $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ were defined to be contracting similarities. Thus $(S_1, \ldots, S_N)$ are contacting similarities on $X \subset \mathbb{R}^d$. Hence, we have

$$\bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(O) \subseteq \bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(X) \quad \text{for all } n .$$

Therefore,

$$\overline{\bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(O)} \subseteq \bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(X) \quad \text{for all } n . \tag{3.4}$$

Thus (3.4) shows that

$$x \in \bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(X) \quad \text{for all } n .$$

Hence

$$x \in \bigcap_n \bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(X) = K_\varnothing ,$$

where the equality $\bigcap_n \bigcup_{|\mathbf{i}|=n} S_{\mathbf{i}}(X) = K_\varnothing$ is well-known (see, e.g. [Fal90, Fal97]). This completes the proof.

2. Analytic approach

Part 1. We first prove that the set

$$K_\varnothing \ \cup \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C)$$

is compact.

For brevity write $L = K_\varnothing \cup \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C)$. Since L is clearly bounded, it suffices to show that L is closed. Therefore, let $x \in \mathbb{R}^d$ and let $(x_n)_n$ be a sequence of points in L such that $x_n \to x$. We must now prove that $x \in L$. We divide the proof of this into two cases.

Case 1: $x_n \in K_\varnothing$ for infinitely many n. In this case there is a subsequence $(x_{n_k})_k$ with $x_{n_k} \in K_\varnothing$ for all k. Since $K_\varnothing$ is closed, we now conclude that $x = \lim_n x_n = \lim_k x_{n_k} \in K_\varnothing \subseteq L$.

Case 2: $x_n \in K_\varnothing$ for finitely many n. Since $x_n \in L = K_\varnothing \cup \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C)$, we conclude that there exists a positive integer n_0 such that

$$x_n \in \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C)$$

for all $n \geq n_0$. Hence, for $n \geq n_0$ we can choose $\mathbf{i}_n \in \Sigma^*$ with

$$x_n \in S_{\mathbf{i}_n}(C) .$$

We now divide Case 2 into two subcases.

Subcase 2.1: $\liminf_n |\mathbf{i}_n| < \infty$. Let $m = \liminf_n |\mathbf{i}_n| \in \mathbb{N}$. Since $|\mathbf{i}_n| \in \mathbb{N}$ for all n, it follows from the definition of m that there is a subsequence $(\mathbf{i}_{n_k})_k$ such that $\{\mathbf{i}_{n_k} \,|\, k \in \mathbb{N}\} \subseteq \Sigma^m$. In particular, this shows that the set $\{\mathbf{i}_{n_k} \,|\, k \in \mathbb{N}\}$ is finite. This implies that there is a string $\mathbf{j} \in \{\mathbf{i}_{n_k} \,|\, k \in \mathbb{N}\}$ and a further subsequence $(\mathbf{i}_{n_{k_l}})_l$ such that $\mathbf{i}_{n_{k_l}} = \mathbf{j}$ for all l. We deduce from this that $x_{n_{k_l}} \in S_{\mathbf{i}_{n_{k_l}}}(C) = S_{\mathbf{j}}(C)$ for all l, and since $S_{\mathbf{j}}(C)$ is closed we therefore conclude that $x = \lim x_n = \lim_l x_{n_{k_l}} \in S_{\mathbf{j}}(C) \subseteq L$.

Subcase 2.2: $\liminf_n |\mathbf{i}_n| = \infty$. In this case we conclude that $|\mathbf{i}_n| \to \infty$ as $n \to \infty$, whence

$$
\begin{aligned}
\operatorname{diam}\big(S_{\mathbf{i}_n}(C) \cup S_{\mathbf{i}_n}(K_\varnothing) \big) &= r_{\mathbf{i}_n} \operatorname{diam}\big(C \cup K_\varnothing \big) \\
&\leq r_{\max}^{|\mathbf{i}_n|} \operatorname{diam}\big(C \cup K_\varnothing \big) \\
&\to 0\,.
\end{aligned}
\tag{3.5}
$$

Next, for each $n \in \mathbb{N}$ choose a point

$$y_n \in S_{\mathbf{i}_n}(K_\varnothing)\,.$$

We will now prove that $(y_n)_n$ is Cauchy. Indeed, for positive integers n and m we have

$$
\begin{aligned}
|y_n - y_m| &\leq |y_n - x_n| + |x_n - x_m| + |x_m - y_m| \\
&\leq \operatorname{diam}\big(S_{\mathbf{i}_n}(C) \cup S_{\mathbf{i}_n}(K_\varnothing) \big) + |x_n - x_m| + \operatorname{diam}\big(S_{\mathbf{i}_m}(C) \cup S_{\mathbf{i}_m}(K_\varnothing) \big)\,.
\end{aligned}
$$

This inequality combined with (3.5) and the fact that $(x_n)_n$ is convergent shows that $(y_n)_n$ is Cauchy.
Since $(y_n)_n$ is Cauchy, we conclude that there is $y \in \mathbb{R}^d$ such that $y_n \to y$. We now observe that

$$x = y\,.$$

Indeed, it follows from (3.5) that

$$
\begin{aligned}
|x - y| &\leq |x - x_n| + |x_n - y_n| + |y_n - y| \\
&\leq |x - x_n| + \operatorname{diam}\big(S_{\mathbf{i}_n}(C) \cup S_{\mathbf{i}_n}(K_\varnothing) \big) + |y_n - y| \\
&\to 0\,.
\end{aligned}
$$

We conclude from this that $x = y$.
Finally, since $y_n \in S_{\mathbf{i}_n}(K_\varnothing) \subseteq K_\varnothing$ and $K_\varnothing$ is closed, we see that $x = y = \lim_n y_n \in K_\varnothing \subseteq L$.

Part 2. Next we show that

$$K_C = K_\varnothing \;\cup\; \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C)\,.$$

For brevity write $L = K_\varnothing \cup \bigcup_{\mathbf{i}} S_{\mathbf{i}}(C)$. Since K_C is the only non-empty and compact set satisfying $K_C = \bigcup_i S_i(K_C) \cup C$, it suffices to show that L is non-empty and compact and satisfies $L = \bigcup_i S_i(L) \cup C$. To prove this first observe that it follows from Part 1 that L is non-empty and compact. Next, we prove that L satisfies $L = \bigcup_i S_i(L) \cup C$. Indeed, we have

$$
\begin{aligned}
\bigcup_i S_i(L) \cup C &= \bigcup_i S_i\left(K_\varnothing \cup \bigcup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}}(C) \right) \cup C \\
&= \bigcup_i S_i(K_\varnothing) \cup \bigcup_i \bigcup_{\mathbf{i} \in \Sigma^*} S_{i\mathbf{i}}(C) \;\cup C \\
&= \bigcup_i S_i(K_\varnothing) \;\cup\; \bigcup_{\mathbf{j} \in \Sigma^*} S_{\mathbf{j}}(C)\,.
\end{aligned}
\tag{3.6}
$$

Finally, since $\cup_i S_i(K_\varnothing) = K_\varnothing$, we conclude from (3.6) that

$$\begin{aligned}\bigcup_i S_i(L) \cup C &= K_\varnothing \cup \bigcup_{\mathbf{j}} S_{\mathbf{j}}(C) \\ &= L\,.\end{aligned}$$

This completes the proof.

Theorem 3.10.
1. We have the following formula for any countably stable dimension of the set K_C *satisfying (2.3).*

$$\dim K_C = \max(\dim K_\varnothing, \dim C)\,. \tag{3.7}$$

In particular, we have the following formulae for the Hausdorff and packing dimensions of the set K_C *satisfying (2.3).*

$$\dim_{\mathrm{H}} K_C = \max(\dim_{\mathrm{H}} K_\varnothing, \dim_{\mathrm{H}} C)\,, \tag{3.8}$$

$$\dim_{\mathrm{P}} K_C = \max(\dim_{\mathrm{P}} K_\varnothing, \dim_{\mathrm{P}} C)\,, \tag{3.9}$$

where $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension and* $\dim_{\mathrm{P}}$ *denotes the packing dimension.*

2. Assume that the sets $(S_1 K_C, \ldots, S_N K_C, C)$ *are pairwise disjoint. We have the following formula for the upper box dimension of the set* K_C *satisfying (2.3).*

$$\overline{\dim}_{\mathrm{B}}(K_C) = \max(\overline{\dim}_{\mathrm{B}}(K_\varnothing), \overline{\dim}_{\mathrm{B}}(C))\,, \tag{3.10}$$

where $\overline{\dim}_{\mathrm{B}}$ *denotes the upper box dimension.*

Proof. 1. It follows from Lemma 3.9 that

$$K_C = K_\varnothing \cup \bigcup_{\mathbf{i}\in\Sigma^*} S_{\mathbf{i}}(C)\,.$$

Thus, using that the Hausdorff and the packing dimensions are countably stable, we obtain (3.8) and (3.9) respectively. More precisely, we have

$$\begin{aligned}\dim_{\mathrm{H}} K_C &= \max\Big(\dim_{\mathrm{H}} K_\varnothing \ , \ \sup_{\mathbf{i}\in\Sigma^*} \dim_{\mathrm{H}} S_{\mathbf{i}} C \Big) \\ &= \max\Big(\dim_{\mathrm{H}} K_\varnothing \ , \ \sup_{\mathbf{i}\in\Sigma^*} \dim_{\mathrm{H}} C \Big) \\ &= \max\Big(\dim_{\mathrm{H}} K_\varnothing \ , \ \dim_{\mathrm{H}} C \Big)\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\dim_{\mathrm{P}} K_C &= \max\Big(\dim_{\mathrm{P}} K_\varnothing \ , \ \sup_{\mathbf{i}\in\Sigma^*} \dim_{\mathrm{P}} S_{\mathbf{i}} C \Big) \\ &= \max\Big(\dim_{\mathrm{P}} K_\varnothing \ , \ \sup_{\mathbf{i}\in\Sigma^*} \dim_{\mathrm{P}} C \Big) \\ &= \max\Big(\dim_{\mathrm{P}} K_\varnothing \ , \ \dim_{\mathrm{P}} C \Big)\,.\end{aligned}$$

2. The formula for the upper box-dimension is obtained using results on L^q spectra of inhomogeneous self-similar measures. Therefore for the proof of (3.10), see Section 4.2.1, where we study L^q spectra of inhomogeneous self-similar measures.

$\square$

Comparison with (homogeneous) self-similar sets.
Comparing Theorem 3.10 with $\dim_{\mathrm{H}} K_{\varnothing} = \dim_{\mathrm{P}} K_{\varnothing} = \dim_{\mathrm{B}} K_{\varnothing} = s$ and Theorem 3.8 we see that Theorem 3.10 provides a natural inhomogeneous extension of the classical homogeneous result in Theorem 3.8. Namely, dimension of the inhomogeneous set equals the maximum of a natural dimension index associated with the homogeneous set and the dimension of the inhomogeneous term.

3.3 Open problems for dimensions of inhomogeneous self-similar sets

Unfortunately, we do not know if a similar result holds for the lower box dimension of K_C. It is natural to ask the following question.

Question 3.11. *Assume that the sets $(S_1 K_C, \ldots, S_N K_C, C)$ are pairwise disjoint. Is it true that*

$$\underline{\dim}_{\mathrm{B}}(K_C) = \max(\underline{\dim}_{\mathrm{B}}(K_{\varnothing}), \underline{\dim}_{\mathrm{B}}(C)), \tag{3.11}$$

where $\underline{\dim}_{\mathrm{B}}$ denotes the lower box-dimension.

It is also quite unsatisfactory that our result for the upper box-dimension is obtained under the assumption that the sets $(S_1 K_C, \ldots, S_N K_C, C)$ are pairwise disjoint. It is natural to ask if the results are true assuming only the appropriate version of the standard Open Set Condition. Namely, assuming Inhomogeneous Open Set Condition (IOSC) which we will state in Section 4.3.1.

Question 3.12. *Are the results in Section 3.2 true if the IOSC is satisfied?*

4 Multifractal analysis of inhomogeneous self-similar measures

4.1 Preliminaries: multifractal analysis

In this section we give a brief introduction to the multifractal analysis. In particular, we will emphasise the importance of multifractal analysis of self-similar measures and their natural generalisations inhomogeneous self-similar measures.

4.1.1 L^q spectra and Rényi dimensions

In section 3.1 we showed how to describe the size of a set by computing its dimension. However, this does not describe the way the measure is allocated within this supporting set. This is done by introducing fractal dimensions of a probability measure. Roughly speaking, there are two types of fractal dimensions of a probability measure. Namely, there are local and global dimensions. For various classes of measures these dimensions are related to each other by multifractal formalism. The global dimensions were essentially introduced by Rényi [Rén60, Rén61] in 1960 as a tool for analyzing various problems in information theory. Indeed, for a probability vector $\mathbf{p} = (p_1, \ldots, p_n)$ and $q \in \mathbb{R}$, Rényi defined the q-entropy $H_{\mathbf{p}}(q)$ of $\mathbf{p}$ by $H_{\mathbf{p}}(q) = \frac{1}{1-q} \log \sum_i p_i^q$ for $q \neq 1$ and $H_{\mathbf{p}}(1) = -\sum_i p_i \log p_i$. Observe that l'Hospital's rule shows that $H_{\mathbf{p}}(q) \to H_{\mathbf{p}}(1)$ as $q \to 1$, and the q-entropies $H_{\mathbf{p}}(q)$ can therefore be regarded as natural generalizations of the usual entropy $H_{\mathbf{p}}(1) = -\sum_i p_i \log p_i$ of $\mathbf{p}$. The entropies $H_{\mathbf{p}}(q)$ are discussed in detail by Rényi in ([Rén70], Chapter 9). In the 1980s Hentshel and Procaccia reintroduced these dimensions to characterize fractals and strange attractors [HP83]. We will now give formal definitions of these global dimensions, namely, we will now formally define closely related L^q spectra and Rényi dimensions.
For a Borel probability measure m on $\mathbb{R}^d$ and $q \in \mathbb{R}$, the lower L^q spectrum $\underline{\tau}_m(q)$ and the upper L^q spectrum $\overline{\tau}_m(q)$ of m are defined as follows. For $q \in \mathbb{R}$ we put

$$\begin{aligned}
\underline{\tau}_m(q) &= \liminf_{r \searrow 0} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x)}{-\log r}, \\
\overline{\tau}_m(q) &= \limsup_{r \searrow 0} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x)}{-\log r},
\end{aligned}$$

where $\operatorname{supp} m$ denotes the support of m.

As it was mentioned above, Rényi dimensions and L^q spectra are closely related. For a Borel probability measure m on $\mathbb{R}^d$ and $q \in [-\infty, \infty]$, the lower and upper q-Rényi dimensions of m are defined by

$$\begin{aligned}
\underline{D}_m(q) &= \liminf_{r \searrow 0} \frac{1}{q-1} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x)}{\log r} \quad \text{for } q \in \mathbb{R} \setminus \{1\}, \\
\overline{D}_m(q) &= \limsup_{r \searrow 0} \frac{1}{q-1} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x)}{\log r} \quad \text{for } q \in \mathbb{R} \setminus \{1\}, \\
\underline{D}_m(1) &= \liminf_{r \searrow 0} \frac{\int_{\operatorname{supp} m} \log m(B(x,r))\, dm(x)}{\log r}, \\
\overline{D}_m(1) &= \limsup_{r \searrow 0} \frac{\int_{\operatorname{supp} m} \log m(B(x,r))\, dm(x)}{\log r},
\end{aligned}$$

and

$$\begin{aligned}
\underline{D}_m(-\infty) &= \liminf_{r\searrow 0} \frac{\log \inf_{x\in \operatorname{supp} m} m(B(x,r))}{\log r}\,,\\
\overline{D}_m(-\infty) &= \limsup_{r\searrow 0} \frac{\log \inf_{x\in \operatorname{supp} m} m(B(x,r))}{\log r}\,,\\
\underline{D}_m(\infty) &= \liminf_{r\searrow 0} \frac{\log \sup_{x\in \operatorname{supp} m} m(B(x,r))}{\log r}\,,\\
\overline{D}_m(\infty) &= \limsup_{r\searrow 0} \frac{\log \sup_{x\in \operatorname{supp} m} m(B(x,r))}{\log r}\,.
\end{aligned}$$

4.1.2 Multifractal spectra

We now turn towards the definition of local dimensions of a probability measure mentioned in the previous section. For a probability measure m on $\mathbb{R}^d$ (or on a general metric space), the lower and upper local dimensions of m at the point x are defined by

$$\begin{aligned}
\underline{\dim}_{\mathrm{loc}}(x;m) &= \liminf_{r\searrow 0} \frac{\log m(B(x,r))}{\log r}\,,\\
\overline{\dim}_{\mathrm{loc}}(x;m) &= \limsup_{r\searrow 0} \frac{\log m(B(x,r))}{\log r}\,.
\end{aligned}$$

If $\underline{\dim}_{\mathrm{loc}}(x;m)$ and $\overline{\dim}_{\mathrm{loc}}(x;m)$ are equal then the common value is called the local dimension of m at the point x and is denoted by $\dim_{\mathrm{loc}}(x;m)$. We define the Hausdorff multifractal spectrum function, $f_{\mathrm{H},m}$, as the Hausdorff dimension of the level sets of the local dimension of m, and we define the packing multifractal spectrum function, $f_{\mathrm{P},m}$ as the packing dimension of the level sets of the local dimension of m, i.e we put

$$\begin{aligned}
f_{\mathrm{H},m}(\alpha) &= \dim_{\mathrm{H}}\left\{ x\in\mathbb{R}^d \;\middle|\; \lim_{r\searrow 0} \frac{\log m(B(x,r))}{\log r} = \alpha \right\}\,,\\
f_{\mathrm{P},m}(\alpha) &= \dim_{\mathrm{P}}\left\{ x\in\mathbb{R}^d \;\middle|\; \lim_{r\searrow 0} \frac{\log m(B(x,r))}{\log r} = \alpha \right\}\,,
\end{aligned}$$

for $\alpha \geq 0$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension and $\dim_{\mathrm{P}}$ denotes the packing dimension.

4.1.3 Multifractal formalism

One of the main significances of the L^q spectra of a measure m, is their relationship with the multifractal spectrum of m, known as multifractal formalism. More precisely, multifractal formalism relates global (L^q spectra) and local (multifractal spectra) behaviours using Legendre transform. Next, recall that the Legendre transform φ^* of a function $\varphi:\mathbb{R}\to\mathbb{R}$ is defined by $\varphi^*(x) = \inf_y(xy+\varphi(y))$. In the 1980s it was conjectured in the physics literature [HJK$^+$86, HP83] that for "good" measures the following result, relating the multifractal spectra functions to the Legendre transform of the L^q spectra, holds: namely (1) that the lower and upper L^q spectra coincide, and (2) that the multifractal spectra functions coincide with the Legendre transform of the L^q spectra. This leads to the following definition.

Definition 4.1. [**The Multifractal Formalism**] *A probability measure m on $\mathbb{R}^d$ is said to satisfy the Multifractal Formalism if*

$$\underline{\tau}_m(q) = \overline{\tau}_m(q)\,,$$

and

$$f_{\mathrm{H},m}(\alpha) = f_{\mathrm{P},m}(\alpha) = \underline{\tau}_m^*(\alpha) = \overline{\tau}_m^*(\alpha)\,,$$

for all $q\in\mathbb{R}$ and all $\alpha\geq 0$.

Nowadays it is well-know that (ordinary) self-similar measures satisfying OSC verify Multifractal Formalism. However, it is easy to find measures that do not satisfy the Multifractal Formalism, and during the 1990s there has therefore been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures for various classes of measures exhibiting some degree of self-similarity, cf. [Fal97] and the references therein.

4.2 L^q spectra and Rényi dimensions of inhomogeneous self-similar measures

L^q spectra and Rényi dimensions of (ordinary) self-similar measures satisfying (2.2) have been studied intensively for the past 15 years and there is a huge body of literature discussing this problem, see, for example, [Fal97] and the references therein. Continuing this line of investigation, in this section we will study the L^q spectra and Rényi dimensions of inhomogeneous self-similar measures. To the best of our knowledge the only results on L^q spectra of inhomogeneous self-similar measures has been obtained by Strichartz in [Str93b] under a number of simplified assumptions. More precisely, Strichartz assumes that the L^q spectra of the condensation measures exist and therefore the renewal type arguments used in [Str93b] to obtain the L^q spectra of the inhomogeneous self-similar measures are fairly straightforward . On the contrary we do not assume that L^q spectra of the condensation measure exists and we develop a general renewal type argument to obtain a formula for the L^q spectra and Renyi dimensions of the inhomogeneous measures. In addition, we give a more comprehensive discussion of the L^q spectra and Renyi dimensions of inhomogeneous measure self-similar measures which includes the study of phase transitions of these measures and applications to the study of the box dimensions of inhomogeneous self-similar sets discussed in Section 3.2 .

4.2.1 L^q spectra: main results and examples

First, recall that r_i denotes the contraction ratio of S_i , and define $\beta : \mathbb{R} \to \mathbb{R}$, by

$$\sum_i p_i^q r_i^{\beta(q)} = 1\,.$$

Observe that the function β is well-defined; indeed, if we let $\varphi_q : \mathbb{R} \to \mathbb{R}$ denote the function $\varphi_q(t) = \sum_i p_i^q r_i^t$, then φ_q is clearly continuous and strictly decreasing with $\lim_{t\to-\infty}\varphi_q(t) = \infty$ and $\lim_{t\to\infty}\varphi_q(t) = 0$, and we can therefore find a unique $\beta(q) \in \mathbb{R}$ such that $\sum_i p_i^q r_i^{\beta(q)} = \varphi_q(\beta(q)) = 1$.
We will now state the first of our main results providing lower and upper bounds for L^q spectra of an inhomogeneous self-similar measure.

Theorem 4.2. *Assume that the sets* $(S_1K_C, \dots, S_NK_C, C)$ *are pairwise disjoint.*

1. *For all* $q \in \mathbb{R}$ *we have*
$$\overline{\tau}_\mu(q) \le \max\Big(\beta(q)\,,\,\overline{\tau}_\nu(q)\Big)\,.$$
2. *For all* $q \in \mathbb{R}$ *we have*
$$\min\Big(\beta(q)\,,\,\underline{\tau}_\nu(q)\Big) \le \underline{\tau}_\mu(q)\,.$$
3. *For all* $q \ge 1$ *we have*
$$\begin{aligned}\max\Big(\beta(q)\,,\,\underline{\tau}_\nu(q)\Big) &\le \underline{\tau}_\mu(q)\,,\\ \max\Big(\beta(q)\,,\,\overline{\tau}_\nu(q)\Big) &\le \overline{\tau}_\mu(q)\,.\end{aligned}$$

The proof of Theorem 4.2 is given in Section 4.2.4. The following exact value for upper L^q spectrum of an inhomogeneous self-similar measure for $q \geq 1$ follows immediately from Theorem 4.2.

Corollary 4.3. *Assume that the sets* $(S_1K_C, \ldots, S_NK_C, C)$ *are pairwise disjoint. For all* $q \geq 1$ *we have*
$$\overline{\tau}_\mu(q) \;=\; \max\Big(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,\Big)\,.$$

Corollary 4.3 gives a formula for the L^q spectra of an inhomogeneous self-similar measure. We now make the following three additional comments related to the result in Corollary 4.3.

(1) Comparison with (homogeneous) self-similar measures.
The L^q spectra, $\underline{\tau}_{\mu_0}(q)$ and $\overline{\tau}_{\mu_0}(q)$, of a (homogeneous) self-similar measure μ_0 satisfying (2.2) have been studied intensively during the past 15 years, cf., for example, the surveys [[Lau95], [Heu07]] or the textbook [Fal97] and the references therein, and it is instructive to compare the results in Theorem 4.2 and Corollary 4.3 with the corresponding results in [Lau95]. We will now state the result in [Lau95].

Theorem 4.4. [See, for example, [Lau95]] *Let* $S_i : \mathbb{R}^d \to \mathbb{R}^d$ *for* $i = 1, \ldots, N$ *be contracting similarities and and write* r_i *for the contracting ratio of* S_i. *Let* $(p_1, \ldots, p_N)$ *be a probability vector and let* μ_0 *be the (homogeneous) self-similar measure satisfying (2.2). Finally, for* $q \in \mathbb{R}$, *let* $\beta_0(q)$ *be defined by*
$$\sum_i p_i^q r_i^{\beta_0(q)} = 1\,.$$
If the Open Set Condition is satisfied, then
$$\underline{\tau}_{\mu_0}(q) = \overline{\tau}_{\mu_0}(q) = \beta_0(q) \tag{4.1}$$
for all q.

Comparing Corollary 4.3 and Theorem 4.4, we see that Corollary 4.3 provides a natural inhomogeneous extension of the classical homogeneous result in Theorem 4.4 . Indeed, this extension is similar to the formulas for the Hausdorff dimension (3.8) and the packing dimension (3.9) of an inhomogeneous self-similar set discussed earlier: namely, the dimension of the inhomogeneous set/measure equals the maximum of a natural dimension index associated with the homogeneous set/measure and the dimension of the inhomogeneous term.

(2) Collapsing of the L^q spectrum of μ. The following rather surprising result follows from Corollary 4.3. Namely, regardless of how the maps $(S_1, \ldots, S_N)$ are chosen and regardless of how the measure ν is chosen, then the L^q spectrum of the inhomogeneous measure μ always collapses and becomes *identical* to that of ν for all q sufficiently close to 1.

Corollary 4.5. *Assume that the sets* $(S_1K_C, \ldots, S_NK_C, C)$ *are pairwise disjoint. Then there exists* $q_0 > 1$ *such that*
$$\overline{\tau}_\mu(q) \;=\; \overline{\tau}_\nu(q)$$
for all $q \in [1, q_0]$.

Proof. Firstly, note that it is well-known (and easily seen) that $\overline{\tau}_\nu$ is convex, and therefore, in particular, continuous. Also observe that $\overline{\tau}_\nu(1) = 0$. Secondly, note that β is continuous with $\beta(1) < 0$. Since the functions $\overline{\tau}_\nu$ and β are continuous with $\beta(1) < \overline{\tau}_\nu(1)$, there exists $q_0 > 1$ such that $\beta(q) < \overline{\tau}_\nu(q)$ for all $q \in [1, q_0]$. Corollary 4.3 therefore implies that $\overline{\tau}_\mu(q) = \max(\beta(q), \overline{\tau}_\nu(q)) = \overline{\tau}_\nu(q)$ for all $q \in [1, q_0]$.

(3) Phase transitions. Another interesting result following from Corollary 4.3 is that an inhomogeneous self-similar measure often has phase transitions. This is in sharp contrast to the behaviour of

(homogeneous) self-similar measures satisfying the Open Set Condition. We will now explain what a phase transition is and prove that inhomogeneous self-similar measures often have phase transitions. Let m be a probability measure on $\mathbb{R}^d$. Due to a formal analogy between the L^q spectra of m and the partition function in statistical mechanics, the L^q spectra $\underline{\tau}_m(q)$ and $\overline{\tau}_m(q)$ are often in the physics literature interpreted as the free energy of "the physical system described by m" as a function of the inverse temperature q. The reader is referred to [[BP97], pp. 128–132;[BS93], pp. 114–126;[Ott93], pp. 309–910] for a discussion of these and other analogies between multifractal analysis and statistical mechanics. In statistical mechanics, phase transitions are manifested as points of non-differentiability of the free energy. The study of the differentiability properties of the L^q spectra $\underline{\tau}_m(q)$ and $\overline{\tau}_m(q)$ can therefore be interpreted as the study of "phase transitions" of the measure m, and following this analogy points q at which one or both of the L^q spectra $\underline{\tau}_m(q)$ and $\overline{\tau}_m(q)$ are non-differentiable are called phase transitions.
It is well known that a (homogeneous) self-similar measure μ_0 satisfying the Open Set Condition does not have any phase transitions. In fact, it follows from Theorem 4.4 that $\underline{\tau}_{\mu_0}(q) = \overline{\tau}_{\mu_0}(q) = \beta_0(q)$ is a real analytic function of q. This is in sharp contrast to the following surprising behaviour of inhomogeneous self-similar measures. Namely, inhomogeneous self-similar measures often have phase transitions. Indeed, the proposition below provides a general condition guaranteeing the existence of phase transitions.

Proposition 4.6. *Assume that the sets* $(S_1K_C, \dots, S_NK_C, C)$ *are pairwise disjoint.*

1. *If there exists* $q_0 \in (1,\infty)$ *such that*
$$\beta(q_0) = \overline{\tau}_\nu(q_0)$$
and $\overline{\tau}_\nu$ *is differentiable at* q_0 *with*
$$\beta'(q_0) \neq \overline{\tau}'_\nu(q_0)\,,$$
then $\overline{\tau}_\mu$ *has a phase transition at* q_0.

2. *In particular, if* $t > 0$ *is a positive real number such that* $\overline{\tau}_\nu(q) = t(1-q)$ *for all* q *(this is, for example, the case if* C *is a* t*-set and* ν *equals the normalized* t*-dimensional Hausdorff measure restricted to* C*) and* $-t < \beta'(1)$ *(this is easily seen to be the case if, for example, the* r_i*'s are sufficiently small), then* $\overline{\tau}_\mu$ *has a phase transition.*

Proof. 1. It follows immediately from the assumptions in Proposition 4.6 and the fact that $\overline{\tau}_\mu(q) = \max(\beta(q), \overline{\tau}_\nu(q))$ for all $q > 1$, that $\overline{\tau}_\mu$ is non-differentiable at q_0.
2. To prove this note that the convexity of β and the fact that $-t < \beta'(1)$, guarantee the existence of a number $q_0 \in (1,\infty)$ such that $\beta(q_0) = t(1-q_0)$ and $\overline{\tau}'_\nu(q_0) = -t < \beta'(1) \le \beta'(q_0)$, i.e. $\beta(q_0) = \overline{\tau}_\nu(q_0)$ and $\overline{\tau}'_\nu(q_0) \neq \beta'(q_0)$. Now Part 1 shows that $\overline{\tau}_\mu$ has a phase transition at q_0. □

Note that the following surprising result follows from Proposition 4.6.(2), namely, even if the inhomogeneous term ν is very well-behaved and the function $\overline{\tau}_\nu$ is real analytic (for example, if ν equals the normalized t-dimensional Hausdorff measure restricted to a t-set C), the resulting inhomogeneous self-similar measure μ can have phase transitions. Below we present examples with several phase transitions.

Example 4.7.

We now present our first concrete example with 2 phase transitions. In this example we take $d=4$ and $N=2$, and we define the maps (S_1,S_2) and the probability vector (p_1,p_2,p) as follows. Let $S_1,S_2:[0,5]^4\to[0,5]^4$ be defined by

$$S_1(x)=\frac{1}{5}x+a_1\,,\quad S_2(x)=\frac{1}{6}x+a_2\,,$$

where $a_i=(2i,2i,2i,2i)$ for $i=1,2$, and let $(p_1,p_2,p)=(\frac{2}{5},\frac{2}{7},\frac{11}{35})$. In this case the function $\beta:\mathbb{R}\to\mathbb{R}$ is given by

$$\left(\frac{2}{5}\right)^q\left(\frac{1}{5}\right)^{\beta(q)}+\left(\frac{2}{7}\right)^q\left(\frac{1}{6}\right)^{\beta(q)}=1\,. \tag{4.2}$$

We now define the measure ν as follows. First, let Σ denote the family of all strings $\mathbf{i}=i_1i_2i_3i_4$ consisting of 4 symbols i_j with $i_j\in\{0,1\}$ for all j. Next, let the maps $T_{\mathbf{i}}:[0,1]^4\to[0,1]^4$ for $\mathbf{i}=i_1i_2i_3i_4\in\Sigma$ be defined by $T_{\mathbf{i}}(x)=\frac{1}{2}x+a_{\mathbf{i}}$ where $a_{\mathbf{i}}=(\frac{i_1}{2},\frac{i_2}{2},\frac{i_3}{2},\frac{i_4}{2})$, and define the probability vector $(p_{\mathbf{i}})_{\mathbf{i}\in\Sigma}$ by $p_{0000}=\frac{71}{100}$ and $p_{\mathbf{i}}=\frac{29}{1500}$ for $\mathbf{i}\in\Sigma\setminus\{0000\}$. We now let ν denote the self-similar measure associated with the list $(T_{\mathbf{i}},p_{\mathbf{i}})_{\mathbf{i}\in\Sigma}$, i.e. ν is the unique probability measure on $[0,1]^4$ such that $\nu=\sum_{\mathbf{i}}p_{\mathbf{i}}\nu\circ T_{\mathbf{i}}^{-1}$. It is well-known (cf. Theorem 4.4 or [Lau95]) that

$$\underline{\tau}_\nu(q)=\overline{\tau}_\nu(q)=\tau(q)$$

for all q where $\tau(q)$ is given by $\sum_{\mathbf{i}}p_{\mathbf{i}}^q(\frac{1}{2})^{\tau(q)}=1$, i.e.

$$\tau(q)=\frac{\log((\frac{71}{100})^q+15(\frac{29}{1500})^q)}{\log 2}\,. \tag{4.3}$$

The graphs of the functions β and $\underline{\tau}_\nu=\overline{\tau}_\nu=\tau$ are sketched in Figure 4.2.1. A standard calculus argument shows that there exist two numbers $q_0\approx 1.2256$ and $q_1\approx 3.1339$ with $1<q_0<q_1$ such that $\beta(q)<\tau(q)$ for $q\notin(q_0,q_1)$ and $\tau(q)<\beta(q)$ for $q\in(q_0,q_1)$. It therefore follows from Corollary 4.3 that

$$\overline{\tau}_\mu(q)=\begin{cases}\tau(q) & \text{for } q\in[1,q_0);\\ \beta(q) & \text{for } q\in[q_0,q_1];\\ \tau(q) & \text{for } q\in(q_1,\infty).\end{cases}$$

Observe that in this example $\overline{\tau}_\mu$ has phase transitions at q_0 and q_1. This completes the example.

Example 4.8.

We now present our next concrete example with three phase transitions. In this example we again take $d=4$ and we consider the following inhomogeneous self-similar measure. Let $P_1,P_2:[0,5]^4\to[0,5]^4$ be contracting similarities defined by

$$P_1(x)=\frac{1}{3}x+b_1\,,\quad P_2(x)=\frac{1}{4}x+b_2\,,$$

where $b_1=(3,0,0,0)$ and $b_2=(0,3,0,0)$. Also, let (ρ_1,ρ_2,ρ) be a probability vector given by $(\rho_1,\rho_2,\rho)=(\frac{18}{40},\frac{15}{40},\frac{7}{40})$. Now, let λ be the inhomogeneous self-similar measure defined by

$$\lambda=\sum_{i=1}^{2}\rho_i\lambda\circ P_i^{-1}\;+\;\rho\mu\,, \tag{4.4}$$

where μ is the inhomogeneous self-similar measure consinered in Example 4.7. In this case it follows from Corollary 4.3 that for $q\geq 1$

$$\overline{\tau}_\lambda(q)\;=\;\max\Big(\beta_\lambda(q)\,,\,\overline{\tau}_\mu(q)\Big)\,,$$

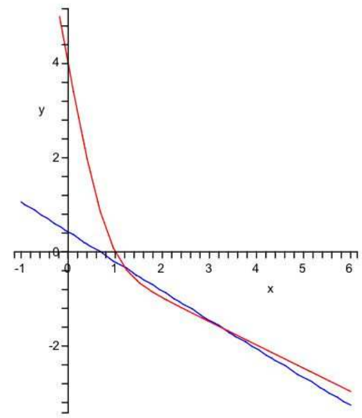

Figure 4.2.1:
This figure shows the graphs of the functions β and $\underline{\tau}_\nu = \overline{\tau}_\nu = \tau$ defined in (4.2) and (4.3), respectively. The graph of β is drawn as a blue line and the graph of $\underline{\tau}_\nu = \overline{\tau}_\nu = \tau$ is drawn as a red line. Observe that $\overline{\tau}_\mu = \max(\beta, \overline{\tau}_\nu)$ has phase transitions at $q_0 \approx 1.2256$ and $q_1 \approx 3.1339$.

with the function $\beta_\lambda : \mathbb{R} \to \mathbb{R}$ given by

$$\left(\frac{18}{40}\right)^q \left(\frac{1}{3}\right)^{\beta_\lambda(q)} + \left(\frac{15}{40}\right)^q \left(\frac{1}{4}\right)^{\beta_\lambda(q)} = 1 . \tag{4.5}$$

From Example 4.7 we have that $\overline{\tau}_\mu(q) = \max\Big(\beta(q)\,,\,\tau(q)\Big)$, for $q \geq 1$. Therefore,

$$\overline{\tau}_\lambda(q) = \max\Big(\beta_\lambda(q)\,,\,\beta(q)\,,\,\tau(q)\Big) .$$

The graphs of the functions $\beta_\lambda\,, \beta(q)$ and $\tau(q)$ are sketched in Figure 4.2.2. As before a standard calculus argument shows that there exist tree numbers $q_0 \approx 1.1545$, $q_1 \approx 1.7401$ and $q_2 \approx 3.1339$ with $1 < q_0 < q_1 < q_2$ such that $\beta(q), \beta_\lambda < \tau(q)$ for $q \in [1, q_0)$, $\tau(q), \beta(q) < \beta_\lambda$ for $q \in [q_0, q_1)$, $\tau(q), \beta_\lambda(q) < \beta$ for $q \in [q_1, q_2)$ and $\beta(q), \beta_\lambda < \tau(q)$ for $q \in [q_2, \infty)$. Thus, we have,

$$\overline{\tau}_\lambda(q) = \begin{cases} \tau(q) & \text{for } q \in [1, q_0); \\ \beta_\lambda(q) & \text{for } q \in [q_0, q_1); \\ \beta(q) & \text{for } q \in [q_1, q_2); \\ \tau(q) & \text{for } q \in [q_2, \infty). \end{cases}$$

Observe that in this example $\overline{\tau}_\lambda$ has phase transitions at q_0 , q_1 , q_2 and therefore we have constructed the inhomogeneous self-similar measure with three phase transitions. This completes the example.

Note that in Example 4.8 we have defined the condensation measure to be itself the inhomogeneous self-similar measure (considered in Example 4.7) with two phase transitions . This way the resulting inhomogeneous self-similar measure had three phase transitions. Obviously, now we can define the condensation measure to be itself the inhomogeneous self-similar measure with three phase transitions (take, for instance, the inhomogeneous self-similar measure in Example 4.8) and obtain the resulting inhomogeneous self-similar measure with four phase transitions. By iterating this process we can therefore construct the inhomogeneous self-similar measure with an arbitrary number of phase transitions.

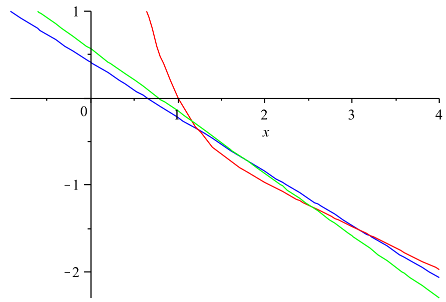

Figure 4.2.2:
This figure shows the graphs of the functions β_λ, β and τ defined in (4.5), (4.2) and (4.3), respectively. The graph of β_λ is drawn as a green line, the graph of β is drawn as a blue line and the graph of τ is drawn as a red line. Observe that $\overline{\tau}_\mu = \max(\beta_\lambda, \beta, \tau)$ has phase transitions at $q_0 \approx 1.1545$, $q_1 \approx 1.7401$ and $q_2 \approx 3.1339$.

For a Borel probability measure m on $\mathbb{R}^d$ and $q \in \mathbb{R}$, we can also define the following variations of the lower and upper L^q spectra of m. Namely, for $q \in \mathbb{R}$ we put

$$\underline{T}_m(q) \quad = \quad \liminf_{r \searrow 0} \frac{\log \frac{1}{r^d} \int m(B(x,r))^q \, d\mathcal{L}^d(x)}{-\log r}, \tag{4.6}$$

$$\overline{T}_m(q) \quad = \quad \limsup_{r \searrow 0} \frac{\log \frac{1}{r^d} \int m(B(x,r))^q \, d\mathcal{L}^d(x)}{-\log r}, \tag{4.7}$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. It is not difficult to see that $\underline{\tau}_m(q) = \underline{T}_m(q)$ and $\overline{\tau}_m(q) = \overline{T}_m(q)$ for $q \geq 1$; however the values may differ for $q < 1$. The dimensions $\underline{T}_\mu(q)$ and $\overline{T}_\mu(q)$ satisfy a relation similar to Corollary 4.3 for $q \geq 0$.

Theorem 4.9. *Assume that the sets $(S_1 K_C, \ldots, S_N K_C, C)$ are pairwise disjoint. For all $0 \leq q$ we have*

$$\overline{T}_\mu(q) \; = \; \max\Big(\beta(q) \, , \, \overline{T}_\nu(q) \Big).$$

The proof of Theorem4.9 is given in Section 4.2.5

As an application of Theorem 4.9 we will obtain a formula for the upper box-dimension of an inhomogeneous self-similar set K_C stated in Theorem 3.10(2).

Proof of Theorem 3.10(2). Momentarily, writing $E = \operatorname{supp} m$ for the support of a probability measure m on $\mathbb{R}^d$ and $B(E,r) = \big\{ x \in \mathbb{R}^d \;\big|\; \operatorname{dist}(x,E) \leq r \big\}$, we see that (cf. [Fal97], p. 20, (2.5))

$$\overline{T}_m(0) = \limsup_{r \searrow 0} \frac{\log \frac{1}{r^d} \int m(B(x,r))^0 \, d\mathcal{L}^d(x)}{-\log r} = \limsup_{r \searrow 0} \frac{\log \frac{1}{r^d} \mathcal{L}^d(B(E,r))}{-\log r} = \overline{\dim}_{\mathrm{B}}(E),$$

where $\overline{\dim}_{\mathrm{B}}(E)$ denotes the upper box-dimension of E. By putting $q = 0$ in Theorem 4.9 , we therefore obtain the result in Theorem 3.10 (2). Namely, we have

$$\overline{\dim}_{\mathrm{B}}(K_C) \; = \; \max\Big(\beta(0) \, , \, \overline{\dim}_{\mathrm{B}}(C) \Big).$$

Since $\overline{\dim}_{\mathrm{B}}(K_\varnothing) = \beta(0)$ (cf. [Fal90] or [Hut81]), this can be written as

$$\overline{\dim}_{\mathrm{B}}(K_C) \;=\; \max\Big(\overline{\dim}_{\mathrm{B}}(K_\varnothing)\,,\, \overline{\dim}_{\mathrm{B}}(C)\Big)\,;$$

recall that $K_\varnothing$ is the self-similar set satisfying $K_\varnothing = \bigcup_i S_i(K_\varnothing)$, cf. (2.1).

4.2.2 Rényi dimensions: main results

Obviously, we immediately obtain the following results for Rényi dimensions from Theorem 4.2 and Corollary 4.3 .

Theorem 4.10. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint.*

1. *For all $q \in \mathbb{R}\setminus\{1\}$ we have*

$$\overline{D}_\mu(q) \;\le\; \max\Big(\frac{\beta(q)}{1-q}\,,\, \overline{D}_\nu(q)\Big)\,.$$

2. *For all $q \in \mathbb{R}\setminus\{1\}$ we have*

$$\min\Big(\frac{\beta(q)}{1-q}\,,\, \underline{D}_\nu(q)\Big) \;\le\; \underline{D}_\mu(q)\,.$$

3. *For all $1 < q$ we have*

$$\begin{aligned} \underline{D}_\mu(q) \quad &\le \min\Big(\tfrac{\beta(q)}{1-q}\,,\, \underline{D}_\nu(q)\Big)\,, \\ \overline{D}_\mu(q) \quad &\le \min\Big(\tfrac{\beta(q)}{1-q}\,,\, \overline{D}_\nu(q)\Big)\,. \end{aligned}$$

Corollary 4.11. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. For all $1 < q$ we have*

$$\underline{D}_\mu(q) \;=\; \min\Big(\frac{\beta(q)}{1-q}\,,\, \underline{D}_\nu(q)\Big)\,.$$

We now show that the result from Corollary 4.11 also holds in the two limiting cases for $q = 1$ and $q = \infty$, and that the result from Theorem 4.10.(1) also holds in the limiting case $q = -\infty$.

Theorem 4.12. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. We have*

$$\underline{D}_\nu(1) \;\le\; \underline{D}_\mu(1) \;\le\; \overline{D}_\mu(1) \;\le\; \overline{D}_\nu(1)\,.$$

In particular, if $\underline{D}_\nu(1) = \overline{D}_\nu(1)$, then

$$\underline{D}_\mu(1) \;=\; \underline{D}_\nu(1)\,.$$

Since clearly $\lim_{q\to1^+} \frac{\beta(q)}{1-q} = \infty$, this may be written as

$$\underline{D}_\mu(1) \;=\; \min\Big(\lim_{q\to1^+}\frac{\beta(q)}{1-q}\,,\, \underline{D}_\nu(1)\Big)\,.$$

Theorem 4.13. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. We have*

$$\underline{D}_\mu(\infty) \;=\; \min\Big(\min_i \frac{\log p_i}{\log r_i}\,,\, \underline{D}_\nu(\infty)\Big)\,.$$

Since clearly $\lim_{q\to\infty} \frac{\beta(q)}{1-q} = \min_i \frac{\log p_i}{\log r_i}$, this may be written as

$$\underline{D}_\mu(\infty) \;=\; \min\Big(\lim_{q\to\infty}\frac{\beta(q)}{1-q}\,,\, \underline{D}_\nu(\infty)\Big)\,.$$

Theorem 4.14. *Assume that the sets* $(S_1K_C, \ldots, S_NK_C, C)$ *are pairwise disjoint. We have*

$$\overline{D}_\mu(-\infty) \le \max\left(\max_i \frac{\log p_i}{\log r_i}, \overline{D}_\nu(-\infty) \right).$$

Since clearly $\lim_{q\to-\infty} \frac{\beta(q)}{1-q} = \max_i \frac{\log p_i}{\log r_i}$ *, this may be written as*

$$\overline{D}_\mu(-\infty) \le \max\left(\lim_{q\to-\infty} \frac{\beta(q)}{1-q}, \overline{D}_\nu(-\infty) \right).$$

Theorem 4.12 is proved in Section 4.2.7, and Theorem 4.13 and Theorem 4.14 are proved in Section 4.2.6.

4.2.3 Proofs of the results for L^q spectra and the Rényi dimensions: a simple general lemma

We now turn towards the proofs of the results in Section 4.2.1 and Section 4.2.2 .
Note that throughout this Section, we will use the notation introduced earlier. Also, write

$$r_{\min} = \min_i r_i \,, \quad r_{\max} = \max_i r_i \,.$$

In this section we will prove a simple and very general lemma. This lemma will be useful for obtaining bounds for the L^q spectrum and the Rényi dimensions in subsequent sections. We first state and prove the lemma. After the statement and the proof of the lemma, we will attempt to provide an explanation of how the lemma is used in the subsequent sections of this part of the thesis.

Lemma 4.15. *Let* $\diamond : (0,\infty)\times(0,\infty) \to (0,\infty)$ *be a commutative and associative binary operation, and assume that if* $x, y, z \in (0,\infty)$ *with* $x \le y$ *, then*

$$x \diamond z \le y \diamond z\,. \tag{4.8}$$

Let $\star : (0,\infty)\times(0,\infty) \to (0,\infty)$ *be a binary operation, and assume that if* $a, x, y \in (0,\infty)$ *with* $x \le y$ *, then*

$$a \star x \le a \star y\,. \tag{4.9}$$

Fix $a_1, \ldots, a_N > 0$ *and a function* $u : (0,\infty) \to (0,\infty)$ *. Let* $r_0 > 0$ *and let* $F, G : (0,\infty) \to \mathbb{R}$ *be two real valued functions. Assume that*

$$\begin{aligned} F(r) &\le \left(\diamond_i \left(a_i \star F\left(\frac{r}{r_i}\right)\right)\right) \diamond u(r)\,, \\ G(r) &\ge \left(\diamond_i \left(a_i \star G\left(\frac{r}{r_i}\right)\right)\right) \diamond u(r)\,, \end{aligned} \tag{4.10}$$

for all $0 < r < r_0$ *. If* $F(r) \le G(r)$ *for all* $r \in [r_{\min} r_0, r_0]$ *, then* $F(r) \le G(r)$ *for all* $r \in (0, r_0]$ *.*

Proof. Assume that $F(r) \le G(r)$ for all $r \in [r_{\min} r_0, r_0]$. We now prove by induction after $n \in \mathbb{N} \cup \{0\}$, that $F(r) \le G(r)$ for all $r \in [r_{\max}^n r_{\min} r_0, r_0]$. The start of the induction follows from the fact that we are assuming that $F(r) \le G(r)$ for all $r \in [r_{\min} r_0, r_0]$. Next, assume that $n \in \{0,1,2,\ldots\}$ and that $F(r) \le G(r)$ for all $r \in [r_{\max}^n r_{\min} r_0, r_0]$. We must now show that $F(r) \le G(r)$ for all $r \in [r_{\max}^{n+1} r_{\max} r_0, r_0]$. Therefore, let $r \in [r_{\max}^{n+1} r_{\min} r_0, r_0]$. If $r \in [r_{\max}^n r_{\min} r_0, r_0]$, then it follows from the inductive hypothesis that $F(r) \le G(r)$. We may thus assume that $r \in [r_{\max}^{n+1} r_{\min} r_0, r_{\max}^n r_{\min} r_0]$. This implies that $\frac{r}{r_i} \le \frac{r_{\max}^n r_{\min} r_0}{r_i} \le r_{\max}^n r_0 \le r_0$ and $\frac{r}{r_i} \ge \frac{r_{\max}^{n+1} r_{\min} r_0}{r_i} \ge r_{\max}^n r_{\min} r_0$ for all i , whence $\frac{r}{r_i} \in [r_{\max}^n r_{\min} r_0, r_0]$ for all i . The inductive hypothesis therefore implies that

$$F\left(\frac{r}{r_i}\right) \le G\left(\frac{r}{r_i}\right),$$

for all i, and so

$$a_i \star F\left(\frac{r}{r_i}\right) \leq a_i \star G\left(\frac{r}{r_i}\right),$$

for all i. Hence

$$\begin{aligned}
F(r) &\leq \left(\diamond_i \left(a_i \star F\left(\frac{r}{r_i}\right)\right)\right) \diamond u(r), \\
&\leq \left(\diamond_i \left(a_i \star G\left(\frac{r}{r_i}\right)\right)\right) \diamond u(r), \\
&\leq G(r).
\end{aligned}$$

This completes the proof. $\square$

We are interested in the following four special cases of Lemma 4.15 .

Case 1: Obtaining bounds for the Rényi dimensions for $q \neq 1, \pm\infty$. In order to obtain bounds for the Rényi dimensions for $q \neq 1, \pm\infty$, we fix $q \in [-\infty, \infty]$ with $q \neq 1, \pm\infty$ and apply Lemma 4.15 to the following setting. Namely, the operations $\diamond$ and $\star$ are defined by $x \diamond y = x + y$ and $a \star x = ax$, and the numbers a_i and the function u are defined by $a_i = p_i^q$ and $u(r) = p^q r^{-t}$ (for an appropriate choice of t). In this case the inequalities in (4.10) become

$$\begin{aligned}
F(r) &\leq \sum_i p_i^q F\left(\frac{r}{r_i}\right) + p^q r^{-t}, \\
G(r) &\geq \sum_i p_i^q G\left(\frac{r}{r_i}\right) + p^q r^{-t}.
\end{aligned} \tag{4.11}$$

Hence, if $F, G : (0,\infty) \to (0,\infty)$ are functions such that (4.11) is satisfied and $F(r) \leq G(r)$ for all $r \in [r_{\min} r_0, r_0]$, then $F(r) \leq G(r)$ for all $r \in (0, r_0]$. In Section 4.2.4 we show that the functions

$$F(r) = \int \mu(B(x,r))^{q-1}\, d\mu(x)$$

and

$$G(r) = c_0 r^{-t}$$

(for appropriate choices of t and c_0) satisfy (4.11) and $F(r) \leq G(r)$ for all $r \in [r_{\min} r_0, r_0]$, and Lemma 4.15 can be applied to give bounds for the Rényi dimensions (and the L^q spectrum) of μ for $q \neq 1, \pm\infty$.

Case 2: Obtaining bounds for the Rényi dimensions for $q = \infty$. In order to obtain bounds for the Rényi dimensions for $q = \infty$ we apply Lemma 4.15 to the following setting. Namely, the operations $\diamond$ and $\star$ are defined by $x \diamond y = \max(x, y)$ and $a \star x = ax$, and the numbers a_i and the function u are defined by $a_i = p_i$ and $u(r) = pr^t$ (for an appropriate choice of t). In this case the inequalities in (4.10) become

$$\begin{aligned}
F(r) &\leq \max\left(\max_i p_i F\left(\frac{r}{r_i}\right), pr^t\right), \\
G(r) &\geq \max\left(\max_i p_i G\left(\frac{r}{r_i}\right), pr^t\right).
\end{aligned} \tag{4.12}$$

Hence , if $F, G : (0,\infty) \to (0,\infty)$ are functions such that (4.12) is satisfied and $F(r) \leq G(r)$ for all $r \in [r_{\min} r_0, r_0]$, then $F(r) \leq G(r)$ for all $r \in (0, r_0]$. In Section 4.2.6 we show that the functions

$$F(r) = \sup_{x \in \operatorname{supp} \mu} \mu(B(x,r))$$

and

$$G(r)=c_0r^t$$

(for appropriate choices of t and c_0) satisfy (4.12) and $F(r)\le G(r)$ for all $r\in[r_{\min}r_0,r_0]$, and Lemma 4.15 can be applied to give bounds for the Rényi dimensions of μ for $q=\infty$.

Case 3: Obtaining bounds for the Rényi dimensions for $q=-\infty$. In order to obtain bounds for the Rényi dimensions for $q=-\infty$ we apply Lemma 4.15 to the following setting. Namely, the operations $\diamond$ and $\star$ are defined by $x\diamond y=\min(x,y)$ and $a\star x=ax$, and the numbers a_i and the function u are defined by $a_i=p_i$ and $u(r)=pr^t$ (for an appropriate choice of t). In this case the inequalities in (4.10) become

$$\begin{aligned} F(r) &\le \min\left(\min_i p_iF\left(\frac{r}{r_i}\right), pr^t\right),\\ G(r) &\ge \min\left(\min_i p_iG\left(\frac{r}{r_i}\right), pr^t\right). \end{aligned} \tag{4.13}$$

Hence , if $F,G:(0,\infty)\to(0,\infty)$ are functions such that (4.13) is satisfied and $F(r)\le G(r)$ for all $r\in[r_{\min}r_0,r_0]$, then $F(r)\le G(r)$ for all $r\in(0,r_0]$. In Section 4.2.6 we show that the functions

$$F(r)=\inf_{x\in\operatorname{supp}\mu}\mu(B(x,r))$$

and

$$G(r)=c_0r^t$$

(for appropriate choices of t and c_0) satisfy (4.13) and $F(r)\le G(r)$ for all $r\in[r_{\min}r_0,r_0]$, and Lemma 4.15 can be applied to give bounds for the Rényi dimensions of μ for $q=-\infty$.

Case 4: Obtaining bounds for the Rényi dimensions for $q=1$. In order to obtain bounds for the Rényi dimensions for $q=1$ we apply Lemma 4.15 to the following setting. Namely, the operations $\diamond$ and $\star$ are given by $x\diamond y=xy$ and $a\star x=x^a$, and the numbers a_i and the function u are given by $a_i=p_i$ and $u(r)=e^sr^{pt}$ where $s=\sum_i p_i\log p_i+p\log p$ (for an appropriate choice of t). In this case the inequalities in (4.10) become

$$\begin{aligned} F(r) &\le e^s\left(\prod_i F\left(\frac{r}{r_i}\right)^{p_i}\right)r^{tp},\\ G(r) &\ge e^s\left(\prod_i G\left(\frac{r}{r_i}\right)^{p_i}\right)r^{tp}. \end{aligned} \tag{4.14}$$

Hence , if $F,G:(0,\infty)\to(0,\infty)$ are functions such that (4.14) are satisfied and $F(r)\le G(r)$ for all $r\in[r_{\min}r_0,r_0]$, then $F(r)\le G(r)$ for all $r\in(0,r_0]$. In Section 4.2.7 we show that the functions

$$F(r)=\exp\int\log\mu(B(x,r))\,d\mu(x)$$

and

$$G(r)=c_0r^t$$

(for appropriate choices of t and c_0) satisfy (4.14) and $F(r)\le G(r)$ for all $r\in[r_{\min}r_0,r_0]$, and Lemma 4.15 can be applied to give bounds for the Rényi dimensions of μ for $q=1$.

4.2.4 Proofs. The case: $q \neq 1, \pm\infty$.

In this section we prove Theorem 4.2. The proof is divided into two parts. Namely, we first apply Lemma 4.15 to prove Theorem 4.2(1) and Theorem 4.2(2). Next, we prove Theorem 4.2(3). For a Borel probability measure m on $\mathbb{R}^d$ and $q \in \mathbb{R}$, write

$$I_m(q;r) = \int\limits_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x)\,.$$

We now present the proof of Theorem 4.2(1) and Theorem 4.2(2). We first derive a functional equation for $I_m(q;r)$; this in done in Proposition 4.16. Next, we use this functional equation and Lemma 4.15 to prove Theorem 4.2.(1) and Theorem 4.2.(2); this is done in Proposition 4.17 and Proposition 4.18.

Proposition 4.16. *Assume that the sets* $(S_1K_C,\ldots,S_NK_C,C)$ *are pairwise disjoint. Let* $q \in \mathbb{R}$. *Then there exists a positive number* $r_0 > 0$ *such that*

$$I_\mu(q;r) = \sum_i p_i^q I_\mu\left(q;\frac{r}{r_i}\right) + p^q I_\nu(q;r) \tag{4.15}$$

for all $0 < r < r_0$.

Proof. Let $r_0 = \min(\min_{i\neq j}\operatorname{dist}(S_iK_C,S_jK_C)\,,\,\min_i \operatorname{dist}(S_iK_C,C)\,)$. Obviously $r_0 > 0$, since the sets $(S_1K_C,\ldots,S_NK_C,C)$ are assumed to be pairwise disjoint and it is a finite collection of sets. It follows from (2.4) that if $0 < r < r_0$, then

$$\begin{aligned} I_\mu(q;r) &= \sum_i p_i \int\limits_{K_C} \mu(B(x,r))^{q-1}\, d(\mu\circ S_i^{-1})(x) + p\int\limits_{K_C} \mu(B(x,r))^{q-1}\, d\nu(x) \\ &= \sum_i p_i \int\limits_{S_iK_C} \mu(B(x,r))^{q-1}\, d(\mu\circ S_i^{-1})(x) + p\int\limits_{C} \mu(B(x,r))^{q-1}\, d\nu(x)\,. \end{aligned} \tag{4.16}$$

However, by using (2.4) once more, we also have for $0 < r < r_0$,

$$\begin{aligned} \mu(B(x,r))^{q-1} &= \left(\sum_i p_i\mu(S_i^{-1}B(x,r)) + p\nu(B(x,r))\right)^{q-1} \\ &= \begin{cases} \left(\,p_i\mu(S_i^{-1}B(x,r))\,\right)^{q-1} & \text{for } x\in S_iK_C; \\ \left(\,p\nu(B(x,r))\,\right)^{q-1} & \text{for } x\in C, \end{cases} \\ &= \begin{cases} p_i^{q-1}\mu(B(S_i^{-1}x,\frac{r}{r_i}))^{q-1} & \text{for } x\in S_iK_C; \\ p^{q-1}\nu(B(x,r))^{q-1} & \text{for } x\in C. \end{cases} \end{aligned} \tag{4.17}$$

Combining (4.16) and (4.17) gives

$$\begin{aligned} I_\mu(q;r) &= \sum_i p_i^q \int\limits_{S_iK_C} \mu\left(B\left(S_i^{-1}x,\frac{r}{r_i}\right)\right)^{q-1} d(\mu\circ S_i^{-1})(x) + p^q\int\limits_C \nu(B(x,r))^{q-1}\, d\nu(x) \\ &= \sum_i p_i^q \int\limits_{K_C} \mu\left(B\left(x,\frac{r}{r_i}\right)\right)^{q-1} d\mu(x) + p^q\int\limits_C \nu(B(x,r))^{q-1}\, d\nu(x) \\ &= \sum_i p_i^q I_\mu\left(q;\frac{r}{r_i}\right) + p^q I_\nu(q;r)\,. \end{aligned}$$

This completes the proof of Proposition 4.16. □

Proposition 4.17. *Assume that the sets* $(S_1K_C, \ldots, S_NK_C, C)$ *are pairwise disjoint. Let* $q \in \mathbb{R}$ *and let* $\max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,) < t$.

1. *There exists a positive number* $r_0 > 0$ *such that*

$$I_\mu(q;r) \le \sum_i p_i^q I_\mu\left(q;\frac{r}{r_i}\right) + p^q r^{-t}$$

for all $0 < r < r_0$.

2. *There exist constants* $r_0, c_0 > 0$ *such that the function* $J : (0,\infty) \to \mathbb{R}$ *defined by* $J(r) = c_0 r^{-t}$ *satisfies*

$$J(r) \ge \sum_i p_i^q J\left(\frac{r}{r_i}\right) + p^q r^{-t}$$

for all $0 < r < r_0$ *, and* $I_\mu(q;r) \le J(r)$ *for all* $r \in [r_{\min} r_0, r_0]$.

3. *We have* $\overline{\tau}_\mu(q) \le \max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,)$.

Proof. 1. Since $\limsup_{r\searrow 0} \frac{\log I_\nu(q;r)}{-\log r} = \overline{\tau}_\nu(q) \le \max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,) < t$, we can find $r_0 \in (0,1)$ such that $\frac{\log I_\nu(q;r)}{-\log r} < t$ for all $0 < r < r_0$, whence $I_\nu(q;r) \le r^{-t}$ for all $0 < r < r_0$. The result follows from this and Proposition 4.16.
2. Let $r_0 > 0$ be as in Part 1. Since $\beta(q) \le \max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,) < t$, we conclude that $\sum_i p_i^q r_i^t < 1$. We can thus choose a constant $c_0 > 0$ such that

$$c_0 \ge \frac{p^q}{1 - \sum_i p_i^q r_i^t}\,, \tag{4.18}$$

and

$$c_0 \ge \frac{\max(\,I_\mu(q;r_0 r_{\min})\,,\,I_\mu(q;r_0)\,)}{\min(\,r_0^{-t}\,,\,(r_0 r_{\min})^{-t}\,)} \tag{4.19}$$

It follows immediately from (4.18) that $c_0 \ge \sum_i p_i^q c_0 r_i^t + p^q$. This clearly implies that the function $J : (0,\infty) \to \mathbb{R}$ defined by $J(r) = c_0 r^{-t}$ satisfies $J(r) \ge \sum_i p_i^q J(\frac{r}{r_i}) + p^q r^{-t}$ for all $0 < r$. Also, it follows from (4.19) that

$$\begin{aligned} I_\mu(q;r) &\le \max(\,I_\mu(q;r_0 r_{\min})\,,\,I_\mu(q;r_0)\,) \\ &\le \frac{\max(\,I_\mu(q;r_0 r_{\min})\,,\,I_\mu(q;r_0)\,)}{\min(\,r_0^{-t}\,,\,(r_0 r_{\min})^{-t}\,)} r^{-t} \\ &\le c_0 r^{-t} = J(r)\,, \end{aligned}$$

for all $r \in [r_{\min} r_0, r_0]$.
3. It follows from Lemma 4.15 (cf., in particular, the discussion in Case 1 following the proof of Lemma 4.15) and Part 2 that $I_\mu(q;r) \le J(r) = c_0 r^{-t}$ for all $0 < r < r_0$. This clearly implies that $\overline{\tau}_\mu(q) \le t$. Since $\max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,) < t$ was arbitrary, we conclude immediately from this that $\overline{\tau}_\mu(q) \le \max(\,\beta(q)\,,\,\overline{\tau}_\nu(q)\,)$. ∎

Remark. The proof of the next Proposition is very similar to the proof of Proposition 4.17 above. However, to clarify this point we will present the proof, but for the rest of this section we will omit presenting such very similar proofs again. We believe that this will ease the exposition of the main ideas of the proofs.

Proposition 4.18. *Assume that the sets* $(S_1K_C,\dots,S_NK_C,C)$ *are pairwise disjoint. Let* $q\in\mathbb{R}$ *and let* $t<\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)$.

1. *There exists a positive number* $r_0>0$ *such that*
$$I_\mu(q;r)\geq\sum_i p_i^q I_\mu\left(q;\frac{r}{r_i}\right)+p^q r^{-t}$$
for all $0<r<r_0$.

2. *There exists constants* $r_0,c_0>0$ *such that the function* $J:(0,\infty)\to\mathbb{R}$ *defined by* $J(r)=c_0r^{-t}$ *satisfies*
$$J(r)\leq\sum_i p_i^q J\left(\frac{r}{r_i}\right)+p^q r^{-t}$$
for all $0<r<r_0$ *, and* $J(r)\leq I_\mu(q;r)$ *for all* $r\in[r_{\min}r_0,r_0]$.

3. *We have* $\underline{\tau}_\mu(q)\geq\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)$.

Proof. 1. Since $\liminf_{r\searrow 0}\frac{\log I_\nu(q;r)}{-\log r}=\underline{\tau}_\nu(q)\geq\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)>t$, we can find $r_0\in(0,1)$ such that $\frac{\log I_\nu(q;r)}{-\log r}>t$ for all $0<r<r_0$, whence $I_\nu(q;r)\geq r^{-t}$ for all $0<r<r_0$. The result follows from this and Proposition 4.16.
2. Let $r_0>0$ be as in Part 1. Since $\beta(q)\geq\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)>t$, we conclude that $\sum_i p_i^q r_i^t>1$. We can thus choose a constant $c_0>0$ such that
$$c_0>0\geq\frac{p^q}{1-\sum_i p_i^q r_i^t}\,,\tag{4.20}$$
and
$$c_0\leq\frac{\min(\,I_\mu(q;r_0r_{\min})\,,\,I_\mu(q;r_0)\,)}{\max(\,r_0^{-t}\,,\,(r_0r_{\min})^{-t}\,)}\tag{4.21}$$
It follows immediately from (4.20) that $c_0\leq\sum_i p_i^q c_0 r_i^t+p^q$. This clearly implies that the function $J:(0,\infty)\to\mathbb{R}$ defined by $J(r)=c_0r^{-t}$ satisfies $J(r)\leq\sum_i p_i^q J(\frac{r}{r_i})+p^q r^{-t}$ for all $0<r<r_0$. Also, it follows from (4.21) that
$$\begin{aligned}I_\mu(q;r)&\geq\min(\,I_\mu(q;r_0r_{\min})\,,\,I_\mu(q;r_0)\,)\\&\geq\frac{\min(\,I_\mu(q;r_0r_{\min})\,,\,I_\mu(q;r_0)\,)}{\max(\,r_0^{-t}\,,\,(r_0r_{\min})^{-t}\,)}r^{-t}\\&\geq c_0r^{-t}=J(r)\,,\end{aligned}$$
for all $r\in[r_{\min}r_0,r_0]$.
3. It follows from Lemma 4.15 (cf., in particular, the discussion in Case 1 following the proof of Lemma 4.15) and Part 2 that $I_\mu(q;r)\geq J(r)=c_0r^{-t}$ for all $0<r<r_0$. This clearly implies that $\underline{\tau}_\mu(q)\geq t$. Since $\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)>t$ was arbitrary, we conclude immediately from this that $\underline{\tau}_\mu(q)\geq\min(\,\beta(q)\,,\,\underline{\tau}_\nu(q)\,)$.

□

The proofs of Proposition 4.17 and Proposition 4.18 complete the proofs of Theorem 4.2(1) and Theorem 4.2(2).
We will now prove Theorem 4.2(3).

Lemma 4.19. *For all* $\mathbf{i}\in\Sigma^*$ *, we have* $\mu(S_{\mathbf{i}}K_C)\geq p_{\mathbf{i}}$.

Proof. Iterating (2.4) we see that

$$\mu = \sum_{|\mathbf{j}|=n} p_{\mathbf{j}}\mu \circ S_{\mathbf{j}}^{-1} + \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}}\nu \circ S_{\mathbf{j}}^{-1}$$

for all n, whence $\mu \geq p_{\mathbf{i}}\mu \circ S_{\mathbf{i}}^{-1}$.
This implies that $\mu(S_{\mathbf{i}}K_C) \geq p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}S_{\mathbf{i}}K_C) \geq p_{\mathbf{i}}\mu(K_C) = p_{\mathbf{i}}$. □

For $r > 0$, write

$$\Gamma(r) = \left\{ \mathbf{i} \in \Sigma^* \,\middle|\, r_{\mathbf{i}} \leq r < r_{\mathbf{i}||\mathbf{i}|-1} \right\}. \tag{4.22}$$

Lemma 4.20. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. Let $q \geq 1$. Then we have*

$$\int_{K_C} \mu(B(x,r))^{q-1}\, d\mu(x) \geq \sum_{\mathbf{i}\in\Gamma(r)} \mu(S_{\mathbf{i}}K_C)^q\,,$$

for all $r > 0$.

Proof. Let $r > 0$. It is clear that $K_C \supseteq \bigcup_{\mathbf{i}\in\Gamma(r)} S_{\mathbf{i}}(K_C)$ and since the sets $(S_{\mathbf{i}}(K_C))_{\mathbf{i}\in\Gamma(r)}$ are pairwise disjoint (because the sets $(S_1K_C, \ldots, S_NK_C, C)$ are assumed to be pairwise disjoint), we conclude that

$$\int_{K_C} \mu(B(x,r))^{q-1}\, d\mu(x) \geq \sum_{\mathbf{i}\in\Gamma(r)} \int_{S_{\mathbf{i}}K_C} \mu(B(x,r))^{q-1}\, d\mu(x)\,. \tag{4.23}$$

Now choose n such that $r_{\max}^n \operatorname{diam}(K_C) \leq r$ (recall, that $r_{\max} = \max_i r_i$). We see from this that $\operatorname{diam}(S_{\mathbf{i}}(K_C)) \leq r_{\max}^n \operatorname{diam}(K_C) \leq r$ for all $\mathbf{i}$ with $|\mathbf{i}| = n$. Hence, if $|\mathbf{i}| = n$ and $x \in S_{\mathbf{i}}K_C$, then $S_{\mathbf{i}}(K_C) \subseteq B(x,r)$, whence

$$\int_{S_{\mathbf{i}}K_C} \mu(B(x,r))^{q-1}\, d\mu(x) \geq \int_{S_{\mathbf{i}}K_C} \mu(S_{\mathbf{i}}K_C)^{q-1}\, d\mu(x) = \mu(S_{\mathbf{i}}K_C)^q\,. \tag{4.24}$$

Combining (4.23) and (4.24) leads to the desired result. □

Proof of Theorem 4.2(3). We must prove that

$$\underline{\tau}_\nu(q) \leq \underline{\tau}_\mu(q)\,, \quad \overline{\tau}_\nu(q) \leq \overline{\tau}_\mu(q)\,, \tag{4.25}$$

and

$$\beta(q) \leq \underline{\tau}_\mu(q)\,. \tag{4.26}$$

We first prove (4.25). Indeed, it follows from (2.3) and (2.4) that $\mu = \sum_i p_i\mu \circ S_i^{-1} + p\nu \geq p\nu$ and that $K_C = \cup_i S_i(K_C) \cup C \supseteq C$. Since $q \geq 1$, we therefore conclude that

$$I_\mu(q;r) = \int_{K_C} \mu(B(x,r))^{q-1}\, d\mu(x) \geq \int_C (p\nu(B(x,r)))^{q-1}\, d(p\nu)(x) = p^q I_\nu(q;r)\,.$$

The inequalities in (4.25) follow immediately from this.
Next, we prove (4.26). Observe that it follows from Lemma 4.19 and Lemma 4.20 that

$$\begin{aligned} I_\mu(q;r) &\geq \sum_{\mathbf{i}\in\Gamma(r)} \mu(S_{\mathbf{i}}K_C)^q \\ &\geq \sum_{\mathbf{i}\in\Gamma(r)} p_{\mathbf{i}}^q\,. \end{aligned} \tag{4.27}$$

Let $K_\varnothing$ be the self-similar set associated with the iterated function system $(S_1,\dots,S_N)$, i.e. $K_\varnothing$ is the unique non-empty and compact set such that $K_\varnothing = \bigcup_i S_i(K_\varnothing)$. Also, let $u_i = p_i^q r_i^{\beta(q)}$. Then $\mathbf{u} = (u_i)_i$ is a probability vector. Let $\mu_\mathbf{u}$ be the self-similar measure associated with the iterated function system $(S_1,\dots,S_N,\mathbf{u})$, i.e. $\mu_\mathbf{u}$ is the unique probability measure such that

$$\mu_\mathbf{u} = \sum_i u_i \mu_\mathbf{u} \circ S_i^{-1}.$$

It is well-known that $\mu_\mathbf{u}(S_\mathbf{i} K_\varnothing) = u_\mathbf{i}$ for all $\mathbf{i} \in \Sigma^*$, cf. [Fal97]. Also, for $\mathbf{i} = i_1 \dots i_n \in \Sigma^*$, write $u_\mathbf{i} = u_{i_1} \cdots u_{i_n}$. It now follows from (4.27) that

$$\begin{aligned} I_\mu(q;r) &\geq \sum_{\mathbf{i}\in\Gamma(r)} p_\mathbf{i}^q \\ &= \sum_{\mathbf{i}\in\Gamma(r)} p_\mathbf{i}^q r_\mathbf{i}^{\beta(q)} r_\mathbf{i}^{-\beta(q)} \\ &= \sum_{\mathbf{i}\in\Gamma(r)} u_\mathbf{i} r_\mathbf{i}^{-\beta(q)} \\ &= \sum_{\mathbf{i}\in\Gamma(r)} \mu_\mathbf{u}(S_\mathbf{i} K_\varnothing) r_\mathbf{i}^{-\beta(q)}. \end{aligned} \tag{4.28}$$

Since clearly $r_\mathbf{i}^{-\beta(q)} \geq c r^{-\beta(q)}$ where $c = r_{\min}^{-\beta(q)}$ for all $\mathbf{i} \in \Gamma(r)$, we deduce from (4.28) that

$$\begin{aligned} I_\mu(q;r) &\geq \sum_{\mathbf{i}\in\Gamma(r)} \mu_\mathbf{u}(S_\mathbf{i} K_\varnothing) r_\mathbf{i}^{-\beta(q)} \\ &\geq c\, r^{-\beta(q)} \sum_{\mathbf{i}\in\Gamma(r)} \mu_\mathbf{u}(S_\mathbf{i} K_\varnothing) \\ &\geq c\, r^{-\beta(q)} \mu_\mathbf{u}\left(\bigcup_{\mathbf{i}\in\Gamma(r)} S_\mathbf{i} K_\varnothing \right) \end{aligned} \tag{4.29}$$

Finally, it is easily seen that $K_\varnothing \subseteq \cup_{\mathbf{i}\in\Gamma(r)} S_\mathbf{i} K_\varnothing$. This and (4.29) imply that

$$I_\mu(q;r) \geq c\, r^{-\beta(q)} \mu_\mathbf{u}\left(\bigcup_{\mathbf{i}\in\Gamma(r)} S_\mathbf{i} K_\varnothing \right) \geq c\, r^{-\beta(q)} \mu_\mathbf{u}(K_\varnothing) = c\, r^{-\beta(q)}.$$

The desired result follows immediately from this by taking logarithms and dividing by $-\log r$. This completes the proof of (4.26).

4.2.5 Proof of Theorem 4.9

In this section we will prove Theorem 4.9. The proof of Theorem 4.9 is very similar to the proof of Theorem 4.2 and will only be sketched, see the remark following Proposition 4.17. For a Borel probability measure m on $\mathbb{R}^d$ and $q \in \mathbb{R}$, write

$$J_m(q;r) = \frac{1}{r^d} \int m(B(x,r))^q \, d\mathcal{L}^d(x).$$

Similarly to the proof of Proposition 4.16 we see that there exists a positive real number $r_0 > 0$ such that

$$J_\mu(q;r) = \sum_i p_i^q J_\mu\left(q; \frac{r}{r_i}\right) + p^q J_\nu(q;r) \tag{4.30}$$

for all $0 < r < r_0$. The proof now proceeds very similarly to the proof of Theorem 4.2 using (4.30) in stead of (4.15).

4.2.6 Proofs. The case: $q = \pm\infty$.

In this section we prove Theorem 4.13 and Theorem 4.14. The proof is divided into two parts. Namely, we first apply Lemma 4.15 to prove that

$$\min\Big(\min_i \frac{\log p_i}{\log r_i}\,,\, \underline{D}_\nu(\infty)\Big) \le \underline{D}_\mu(\infty)\,, \tag{4.31}$$

$$\overline{D}_\mu(-\infty) \le \max\Big(\max_i \frac{\log p_i}{\log r_i}\,,\, \overline{D}_\nu(-\infty)\Big)\,. \tag{4.32}$$

Next, we prove that

$$\underline{D}_\mu(\infty) \le \min\Big(\min_i \frac{\log p_i}{\log r_i}\,,\, \underline{D}_\nu(\infty)\Big)\,. \tag{4.33}$$

For a Borel probability measure m on $\mathbb{R}^d$ write

$$\begin{aligned} I_m(\infty;r) &= \sup_{x\in\operatorname{supp} m} m(B(x,r))\,,\\ I_m(-\infty;r) &= \inf_{x\in\operatorname{supp} m} m(B(x,r))\,. \end{aligned}$$

We now present the proof of (4.31) and (4.32). We first derive a functional equation for $I_m(\pm\infty;r)$; this in done in Proposition 4.21. Next, we use this functional equation and Lemma 4.15 to prove (4.31),(4.32); this is done in Proposition 4.22 and Proposition 4.23 .

Proposition 4.21. *Assume that the sets* $(S_1K_C,\dots,S_NK_C,C)$ *are pairwise disjoint. Then there exists a positive number* $r_0>0$ *such that*

$$I_\mu(\infty;r) = \max\left(\max_i p_i I_\mu\left(\infty;\frac{r}{r_i}\right)\quad,\ pI_\nu(\infty;r)\ \right)\,, \tag{4.34}$$

$$I_\mu(-\infty;r) = \min\left(\min_i p_i I_\mu\left(-\infty;\frac{r}{r_i}\right)\ ,\ pI_\nu(-\infty;r)\right)\,. \tag{4.35}$$

for all $0<r<r_0$.

Proof. Let $r_0 = \min(\,\min_{i\neq j}\operatorname{dist}(S_iK_C,S_jK_C)\,,\min_i\operatorname{dist}(S_iK_C,C)\,)$. It follows by an argument similar to the proof of (4.17) that if $0<r<r_0$, then

$$\mu(B(x,r)) = \begin{cases} p_i\mu(B(S_i^{-1}x,\frac{r}{r_i}) & \text{for } x\in S_iK_C;\\ p\nu(B(x,r) & \text{for } x\in C.\end{cases}$$

Hence

$$\begin{aligned} I_\mu(\infty;r) &= \max\left(\max_i \sup_{x\in S_iK_C}\mu(B(x,r))\,,\,\sup_{x\in C}\mu(B(x,r))\right)\\ &= \max\left(\max_i \sup_{x\in S_iK_C} p_i\mu\left(B\left(S_i^{-1}x,\frac{r}{r_i}\right)\right)\,,\,p\sup_{x\in C}\nu(B(x,r))\right)\\ &= \max\left(\max_i \sup_{x\in K_C} p_i\mu\left(B\left(x,\frac{r}{r_i}\right)\right)\,,\,p\sup_{x\in C}\nu(B(x,r))\right)\\ &= \max\left(\max_i p_i I_\mu\left(\infty;\frac{r}{r_i}\right)\,,\,pI_\nu(\infty;r)\right)\,. \end{aligned}$$

This proves (4.34). Equality (4.35) is proved very similarly (see the remark following Proposition 4.17). This completes the proof of Proposition 4.21 . ∎

Proposition 4.22. *Assume that the sets* $(S_1K_C,\dots,S_NK_C,C)$ *are pairwise disjoint. Let* $t<\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)$.

1. *There exists a positive number* $r_0>0$ *such that*

$$I_\mu(\infty;r)\le\max\left(\max_i p_iI_\mu\left(\infty;\frac{r}{r_i}\right)\,,\,pr^t\right)$$

   for all $0<r<r_0$.

2. *There exists constants* $r_0,c_0>0$ *such that the function* $J:(0,\infty)\to\mathbb{R}$ *defined by* $J(r)=c_0r^t$ *satisfies*

$$J(r)\ge\max\left(\max_i p_iJ\left(\frac{r}{r_i}\right)\,,\,pr^t\right)$$

   for all $0<r<r_0$, *and* $I_\mu(\infty;r)\le J(r)$ *for all* $r\in[r_{\min}r_0,r_0]$.

3. *We have* $\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)\le\underline{D}_\mu(\infty)$.

Proof. 1. Since $\liminf_{r\searrow0}\frac{\log I_\nu(\infty;r)}{\log r}=\underline{D}_\nu(\infty)\ge\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)>t$, we can find $r_0\in(0,1)$ such that $\frac{\log I_\nu(\infty;r)}{\log r}>t$ for all $0<r<r_0$, whence $I_\nu(\infty;r)\le r^t$ for all $0<r<r_0$. The result follows from this and Proposition 4.21 .
2. Let $r_0>0$ be as in Part 1. Choose any $c_0>0$ such that $c_0\ge p$ and $c_0\ge\frac{I_\mu(\infty;r_0)}{(r_{\min}r_0)^t}$, and define $J:(0,\infty)\to\mathbb{R}$ by $J(r)=c_0r^t$. Since $t<\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)\le\min_i\frac{\log p_i}{\log r_i}$, we conclude that $\max_i\frac{p_i}{r_i^t}<1$. This and the fact that $c_0\ge p$, imply that $1\ge\max(\max_i\frac{p_i}{r_i^t}\,,\,\frac{p}{c_0}\,)$. It follows immediately from this inequality that $J(r)\ge\max(\max_i p_iJ(\frac{r}{r_i})\,,\,pr^t\,)$ for all $r>0$. Also, since $c_0\ge\frac{I_\mu(\infty;r_0)}{(r_{\min}r_0)^t}$, it follows that

$$I_\mu(\infty;r)\le I_\mu(\infty;r_0)\le\frac{I_\mu(\infty;r_0)}{(r_{\min}r_0)^t}r^t\le c_0r^t=J(r)\,,$$

for all $r\in[r_{\min}r_0,r_0]$.
3. It follows from Lemma 4.15 (cf; in particular, the discussions in Case 2 and Case 3 following the proof of Lemma 4.15) and Part 2 that $I_\mu(\infty;r)\le J(r)=c_0r^t$ for all $0<r<r_0$. This clearly implies that $t\le\underline{D}_\mu(\infty)$. Since $t<\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)$ was arbitrary, we conclude from this that $\min(\min_i\frac{\log p_i}{\log r_i}\,,\,\underline{D}_\nu(\infty)\,)\le\underline{D}_\mu(\infty)$. □

Proposition 4.23. *Assume that the sets* $(S_1K_C,\dots,S_NK_C,C)$ *are pairwise disjoint. Let* $\max(\max_i\frac{\log p_i}{\log r_i}\,,\,\overline{D}_\nu(-\infty)\,)<t$.

1. *There exists a positive number* $r_0>0$ *such that*

$$I_\mu(-\infty;r)\ge\min\left(\min_i p_iI_\mu\left(-\infty;\frac{r}{r_i}\right)\,,\,pr^t\right)$$

   for all $0<r<r_0$.

2. *There exists constants* $r_0,c_0>0$ *such that the function* $J:(0,\infty)\to\mathbb{R}$ *defined by* $J(r)=c_0r^t$ *satisfies*

$$J(r)\le\min\left(\min_i p_iJ\left(\frac{r}{r_i}\right)\,,\,pr^t\right)$$

   for all $0<r<r_0$, *and* $I_\mu(-\infty;r)\ge J(r)$ *for all* $r\in[r_{\min}r_0,r_0]$.

3. *We have* $\overline{D}_\mu(-\infty) \le \max(\max_i \frac{\log p_i}{\log r_i}, \overline{D}_\nu(-\infty))$.

Proof. The proof is very similar to the proof of Proposition 4.22 and is therefore omitted (see the remark following Proposition 4.17). $\square$

We now turn towards the proof of (4.33).

Proposition 4.24. *Assume that the sets* $(S_1 K_C, \dots, S_N K_C, C)$ *are pairwise disjoint. Then*

$$\underline{D}_\mu(\infty) \le \min\left(\min_i \frac{\log p_i}{\log r_i}, \underline{D}_\nu(\infty)\right).$$

Proof. We must prove that $\underline{D}_\mu(\infty) \le \underline{D}_\nu(\infty)$ and that $\underline{D}_\mu(\infty) \le \min_i \frac{\log p_i}{\log r_i}$.
We first prove that $\underline{D}_\mu(\infty) \le \underline{D}_\nu(\infty)$.
Indeed, since $\mu(B(x,r)) = \sum_i p_i \mu(S_i^{-1}(B(x,r))) + p\nu(B(x,r)) \ge p\nu(B(x,r))$ for all x and all $r > 0$ we see that $I_\mu(\infty;r) \ge pI_\nu(\infty;r)$ for all $r > 0$. This clearly implies that $\underline{D}_\mu(\infty) \le \underline{D}_\nu(\infty)$.
Next, we prove that $\underline{D}_\mu(\infty) \le \min_i \frac{\log p_i}{\log r_i}$. Fix $r > 0$ and write $D = \operatorname{diam}(K_C)$. We claim that

$$\sup_{\mathbf{i}\in\Gamma(r)} p_{\mathbf{i}} \le I_\mu(\infty;rD) \tag{4.36}$$

(recall that $\Gamma(r)$ is defined in (4.22)). To prove (4.36), let $\mathbf{i} \in \Gamma(r)$. Now choose $x \in S_{\mathbf{i}} K_C$. Since $\mathbf{i} \in \Gamma(r)$, we have $r_{\mathbf{i}} \le r$, and so $\operatorname{diam}(S_{\mathbf{i}} K_C) = r_{\mathbf{i}} \operatorname{diam}(K_C) = r_{\mathbf{i}} D \le rD$. We therefore we conclude that $S_{\mathbf{i}} K_C \subseteq B(x,rD)$. We conclude from this that $\mu(S_{\mathbf{i}} K_C) \le \mu(B(x,rD)) \le I_\mu(\infty;rD)$. Taking supremum over all $\mathbf{i} \in \Gamma(r)$ gives

$$\sup_{\mathbf{i}\in\Gamma(r)} \mu(S_{\mathbf{i}} K_C) \le I_\mu(\infty;rD). \tag{4.37}$$

Finally, using the fact that $p_{\mathbf{i}} \le \mu(S_{\mathbf{i}} K_C)$ (by Lemma 4.19), it follows from (4.37) that $\sup_{\mathbf{i}\in\Gamma(r)} p_{\mathbf{i}} \le \sup_{\mathbf{i}\in\Gamma(r)} \mu(S_{\mathbf{i}} K_C) \le I_\mu(\infty;rD)$. This proves (4.36).
Using (4.36), we see that

$$\frac{\log I_\mu(\infty;rD)}{\log r} \le \frac{\log \sup_{\mathbf{i}\in\Gamma(r)} p_{\mathbf{i}}}{\log r} \le \inf_{\mathbf{i}\in\Gamma(r)} \frac{\log p_{\mathbf{i}}}{\log r} \tag{4.38}$$

for all $0 < r < 1$. However, if $\mathbf{i} \in \Gamma(r)$, then $r < r_{\mathbf{i}|(|\mathbf{i}|-1)} \le \frac{r_{\mathbf{i}}}{r_{\min}}$, and so $\frac{\log p_{\mathbf{i}}}{\log r} \le \frac{\log p_{\mathbf{i}}}{\log \frac{r_{\mathbf{i}}}{r_{\min}}}$ for all $\mathbf{i} \in \Gamma(r)$. This and (4.38) imply that

$$\frac{\log I_\mu(\infty;rD)}{\log r} \le \inf_{\mathbf{i}\in\Gamma(r)} \frac{\log p_{\mathbf{i}}}{\log \frac{r_{\mathbf{i}}}{r_{\min}}} = \inf_{\mathbf{i}\in\Gamma(r)} \frac{1}{1 - \frac{\log r_{\min}}{\log r_{\mathbf{i}}}} \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \tag{4.39}$$

for all $0 < r < 1$. Also, if $\mathbf{i} \in \Gamma(r)$, then $r_{\mathbf{i}} \le r$, and so $\frac{1}{1-\frac{\log r_{\min}}{\log r_{\mathbf{i}}}} \le \frac{1}{1-\frac{\log r_{\min}}{\log r}}$. Using this and (4.39), we see that

$$\frac{\log I_\mu(\infty;rD)}{\log r} \le \frac{1}{1 - \frac{\log r_{\min}}{\log r}} \inf_{\mathbf{i}\in\Gamma(r)} \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \tag{4.40}$$

for all $0 < r < 1$.
Next, fix $r \in (0,1)$. Choose $i_0 \in \{1,\dots,N\}$ such that $\frac{\log p_{i_0}}{\log r_{i_0}} = \min_i \frac{\log p_i}{\log r_i}$, and choose $n_r \in \mathbb{N}$ such that $r_{i_0}^{n_r} \le r < r_{i_0}^{n_r - 1}$. Then clearly $\mathbf{i}_r = \underbrace{i_0 \dots i_0}_{n_r \text{ times}} \in \Gamma(r)$. We therefore conclude from (4.40) that

$$\frac{\log I_\mu(\infty;rD)}{\log r} \le \frac{1}{1 - \frac{\log r_{\min}}{\log r}} \inf_{\mathbf{i}\in\Gamma(r)} \frac{\log p_{\mathbf{i}}}{\log r_{\mathbf{i}}} \le \frac{1}{1 - \frac{\log r_{\min}}{\log r}} \frac{\log p_{\mathbf{i}_r}}{\log r_{\mathbf{i}_r}}$$

$$
\begin{aligned}
&= \frac{1}{1-\frac{\log r_{\min}}{\log r}} \frac{\log p_{i_0}^{n_r}}{\log r_{i_0}^{n_r}} = \frac{1}{1-\frac{\log r_{\min}}{\log r}} \frac{\log p_{i_0}}{\log r_{i_0}} \\
&= \frac{1}{1-\frac{\log r_{\min}}{\log r}} \min_i \frac{\log p_i}{\log r_i}
\end{aligned}
$$

for all $0<r<1$. It follows immediately from this inequality that $\underline{D}_\mu(\infty) = \liminf_{r\searrow 0} \frac{\log I_\mu(\infty;r)}{\log r} \le \min_i \frac{\log p_i}{\log r_i}$. □

4.2.7 Proofs. The case: $q=1$.

In this section we prove Theorem 4.12. For a Borel probability measure m on $\mathbb{R}^d$, write

$$
I_m(1;r) = \exp \int_{\operatorname{supp} m} \log m(B(x,r))\, dm(x)
$$

We first derive a functional equation for $I_m(1;r)$; this in done in Proposition 4.25 . Next, we use this functional equation and Lemma 4.15 to Theorem 4.12; this is done in Proposition 4.26 and Proposition 4.27. Throughout this section we write $s = \sum_i p_i \log p_i + p \log p$.

Proposition 4.25. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. Then there exists a positive number $r_0>0$ such that*

$$
I_\mu(1;r) = e^s \left(\prod_i I_\mu\left(1; \frac{r}{r_i}\right)^{p_i} \right) I_\nu(1;r)^p
$$

for all $0<r<r_0$.

Proof. Let $r_0 = \min(\min_{i\neq j} \operatorname{dist}(S_iK_C, S_jK_C), \min_i \operatorname{dist}(S_iK_C, C))$. It follows from (2.4) that if $0<r<r_0$, then

$$
\begin{aligned}
\log I_\mu(1;r) &= \sum_i p_i \int_{K_C} \log \mu(B(x,r))\, d(\mu\circ S_i^{-1})(x) + p \int_{K_C} \log \mu(B(x,r))\, d\nu(x) \\
&= \sum_i p_i \int_{S_iK_C} \log \mu(B(x,r))\, d(\mu\circ S_i^{-1})(x) + p \int_{C} \log \mu(B(x,r))\, d\nu(x) . \qquad (4.41)
\end{aligned}
$$

It follows by an argument similar to the proof of (4.17) that if $0<r<r_0$, then

$$
\log \mu(B(x,r)) = \begin{cases} \log\left(p_i \mu(B(S_i^{-1}x, \frac{r}{r_i}))\right) & \text{for } x \in S_iK_C; \\ \log\left(p\nu(B(x,r))\right) & \text{for } x \in C. \end{cases} \qquad (4.42)
$$

Combining (4.41) and (4.42) gives

$$
\begin{aligned}
\log I_\mu(1;r) &= \sum_i p_i \int_{S_iK_C} \log\left(p_i \mu\left(B\left(S_i^{-1}x, \frac{r}{r_i}\right)\right)\right) d(\mu\circ S_i^{-1})(x) + p \int_C \log\left(p\nu(B(x,r))\right) d\nu(x) \\
&= \sum_i p_i \int_{K_C} \log\left(p_i \mu\left(B\left(x, \frac{r}{r_i}\right)\right)\right) d\mu(x) + p \int_C \log\left(p\nu(B(x,r))\right) d\nu(x) \\
&= \sum_i p_i \log p_i + p \log p + \sum_i p_i \log I_\mu\left(1; \frac{r}{r_i}\right) + p \log I_\nu(1;r) .
\end{aligned}
$$

This completes the proof of Proposition 4.25 . □

Proposition 4.26. *Assume that the sets* $(S_1 K_C, \ldots, S_N K_C, C)$ *are pairwise disjoint. Let* $t < \underline{D}_\nu(1)$.

1. *There exists a positive number* $r_0 > 0$ *such that*

$$I_\mu(1;r) \leq e^s \left(\prod_i I_\mu\left(1; \frac{r}{r_i}\right)^{p_i} \right) r^{pt}$$

for all $0 < r < r_0$.

2. *There exists constants* $r_0, c_0 > 0$ *such that the function* $J : (0,\infty) \to \mathbb{R}$ *defined by* $J(r) = c_0 r^t$ *satisfies*

$$J(r) \geq e^s \left(\prod_i J\left(\frac{r}{r_i}\right)^{p_i} \right) r^{pt}$$

for all $0 < r < r_0$, *and* $I_\mu(1;r) \leq J(r)$ *for all* $r \in [r_{\min} r_0, r_0]$.

3. *We have* $\underline{D}_\nu(1) \leq \underline{D}_\mu(1)$.

Proof. 1. Since $\liminf_{r \searrow 0} \frac{\log I_\nu(1;r)}{\log r} = \underline{D}_\nu(1) > t$, we can find $r_0 \in (0,1)$ such that $\frac{\log I_\nu(1;r)}{\log r} > t$ for all $0 < r < r_0$, whence $I_\nu(1;r) \leq r^t$ for all $0 < r < r_0$. The result follows from this and Proposition 4.25.
2. Let $r_0 > 0$ be as in Part 1. We can clearly choose a constant $c_0 > 0$ such that

$$c_0^p \geq \frac{e^s}{\prod_i r_i^{p_i t}} , \tag{4.43}$$

and

$$c_0 \geq \frac{I_\mu(1;r_0)}{\min(r_0^t \, , \, (r_0 r_{\min})^t)} \tag{4.44}$$

It follows easily from (4.43) that the function $J : (0,\infty) \to \mathbb{R}$ defined by $J(r) = c_0 r^t$ satisfies $J(r) = c_0 r^t = c_0^{\sum_i p_i + p} r^t = (\prod_i c_0^{p_i})\, c_0^p r^t \geq (\prod_i c_0^{p_i}) \left(\frac{e^s}{\prod_i r_i^{p_i t}} \right) r^t = e^s \left(\prod_i \frac{c_0^{p_i} r^{p_i t}}{r_i^{p_i t}} \right) r^{pt} = e^s (\prod_i J(\frac{r}{r_i})^{p_i}) r^{pt}$ for all $0 < r < r_0$. Also, it follows from (4.44) that

$$I_\mu(1;r) \leq I_\mu(1;r_0) \leq \frac{I_\mu(1;r_0)}{\min(r_0^t \, , \, (r_0 r_{\min})^t)} r^t \leq c_0 r^t = J(r) ,$$

for all $r \in [r_{\min} r_0, r_0]$.
3. It follows from Lemma 4.15 (cf; in particular, the discussion in Case 4 following the proof of Lemma 4.15) and Part 2 that $I_\mu(1;r) \leq J(r) = c_0 r^t$ for all $0 < r < r_0$. This clearly implies that $t \leq \underline{D}_\mu(1)$. Since $t < \underline{D}_\nu(1)$ was arbitrary, we conclude from this that $\underline{D}_\nu(1) \leq \underline{D}_\mu(1)$. $\square$

Proposition 4.27. *Assume that the sets* $(S_1 K_C, \ldots, S_N K_C, C)$ *are pairwise disjoint. Let* $\overline{D}_\nu(1) < t$.

1. *There exists a positive number* $r_0 > 0$ *such that*

$$I_\mu(1;r) \geq e^s \left(\prod_i I_\mu\left(1; \frac{r}{r_i}\right)^{p_i} \right) r^{pt}$$

for all $0 < r < r_0$.

2. *There exists constants* $r_0, c_0 > 0$ *such that the function* $J : (0,\infty) \to \mathbb{R}$ *defined by* $J(r) = c_0 r^t$ *satisfies*
$$J(r) \leq e^s \left(\prod_i J\left(\frac{r}{r_i}\right)^{p_i} \right) r^{pt}$$
for all $0 < r < r_0$ *, and* $I_\mu(1;r) \geq J(r)$ *for all* $r \in [r_{\min} r_0, r_0]$ *.*

3. *We have* $\overline{D}_\mu(1) \leq \overline{D}_\nu(1)$ *.*

Proof. The proof is very similar to the proof of Proposition 4.26 and is therefore omitted (see the remark following Proposition 4.17). □

4.3 Multifractal spectra of inhomogeneous self-similar measures

4.3.1 Main results

In this section we compute the multifractal spectra of inhomogeneous self-similar measures. The multifractal spectra are in most cases very difficult to compute, and during the 1990s there has been an enormous interest in computing the multifractal spectra of measures for various classes of measures exhibiting some degree of self-similarity, cf. [Fal97] and the references therein. In particular, the multifractal spectra of self-similar measures satisfying the Open Set Condition has been computed. Recall that the Open Set Condition is defined as follows.

The open set condition (OSC). *The list* $(S_1, \ldots, S_N)$ *is said to satisfy the Open Set Condition, if there is a non-empty, bounded and open set* U *satisfying*

1. *For all* $i = 1, \ldots, N$ *, we have* $S_i U \subseteq U$ *;*
2. *For all* $i, j = 1, \ldots, N$ *with* $i \neq j$ *, we have* $S_i U \cap S_j U = \varnothing$ *;*

The purpose of this section is to compute the multifractal spectra of μ assuming the appropriate inhomogeneous version of the OSC. This version is defined as follows.

The inhomogeneous open set condition (IOSC). *The list* $(S_1, \ldots, S_N, C)$ *is said to satisfy the Inhomogeneous Open Set Condition, if there is a non-empty, bounded and open set* U *satisfying*

1. *For all* $i = 1, \ldots, N$ *, we have* $S_i U \subseteq U$ *;*
2. *For all* $i, j = 1, \ldots, N$ *with* $i \neq j$ *, we have* $S_i U \cap S_j U = \varnothing$ *;*
3. $U \cap K_\varnothing \neq \varnothing$ *;*
4. $C \subseteq \overline{U}$ *;*
5. *For all* $i = 1, \ldots, N$ *, we have* $\dim_{\mathrm{H}}(\overline{S_i U} \cap C) = \dim_{\mathrm{P}}(\overline{S_i U} \cap C) = 0$ *and* $\nu(\overline{S_i U} \cap C) = 0$ *;*
6. *We have* $\dim_{\mathrm{H}}(\partial U \cap C) = \dim_{\mathrm{P}}(\partial U \cap C) = 0$ *and* $\nu(\partial U \cap C) = 0$ *.*

Conditions 1 and 2 are simply the usual Open Set Condition (OSC) for (ordinary) self-similar sets guaranteeing that the overlaps $\overline{S_i U} \cap \overline{S_j U}$ originating from the non-inhomogeneous part of the construction are small for all $i \neq j$. Similarly, Conditions 5 and 6 guarantee that the overlaps $\overline{S_i U} \cap C$ and $\partial U \cap C$ originating from the inhomogeneous part of the construction are small for all i.
We can now state the main result in this section, namely, Theorem 4.28 providing a formula for the multifractal spectra of the inhomogeneous measure μ assuming the IOSC. As in the previous

Section we define $\beta:\mathbb{R}\to\mathbb{R}$ by $\sum_i p_i^q r_i^{\beta(q)}=1$ and recall that β^* denotes the Legendre transform of β.

Theorem 4.28. *Assume that the IOSC is satisfied. Then*

$$\begin{aligned} f_{\mathrm{H},\mu}(\alpha) &= \max\Big(\beta^*(\alpha)\,,\,f_{\mathrm{H},\nu}(\alpha)\Big)\,,\\ f_{\mathrm{P},\mu}(\alpha) &= \max\Big(\beta^*(\alpha)\,,\,f_{\mathrm{P},\nu}(\alpha)\Big)\,,\end{aligned}$$

for all $\alpha\geq 0$.

The proof of Theorem 4.28 is given in Section 4.3.3, Section 4.3.4, Section 4.3.5 and Section 4.3.6. For the benefit of the reader, we will now give a brief overview of the proof of Theorem 4.28. After the overview we make several remarks related to Theorem 4.28.

Brief overview of the proof of Theorem 4.28. The proof of Theorem 4.28 is divided into 4 parts as follows:

Part 1: Section 4.3.3. In Section 4.3.3 we prove (see Proposition 4.44) a useful auxiliary result, namely, if U is the open set in the IOSC, then

$$\mu(S_{\mathbf{i}}K_C)=\mu(\overline{S_{\mathbf{i}}U})=p_{\mathbf{i}}$$

for all $\mathbf{i}\in\Sigma^*$ (recall that $S_{\mathbf{i}}=S_{i_1}\circ\cdots\circ S_{i_n}$ and $p_{\mathbf{i}}=p_{i_1}\cdots p_{i_n}$ for $\mathbf{i}=i_1\ldots i_n\in\Sigma^*$). This result plays an important role through out the remaining parts of the proof.

Part 2: Section 4.3.4. In Section 4.3.4 we relate the multifractal spectra of μ to the multifractal spectra of ν. More precisely, we first prove Lemma 4.46 which states that if U is the open set in the IOSC, then

$$\begin{aligned}\Big\{x\in K_C\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\} &= \Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\\ &\cup\bigcup_{\mathbf{i}\in\Sigma^*}S_{\mathbf{i}}\left\{x\in C\setminus\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(x;\nu)=\alpha\right\}\\ &\cup\bigcup_{\mathbf{i}\in\Sigma^*}S_{\mathbf{i}}\left\{x\in C\cap\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\right\}.\end{aligned}$$

The following relationship between the multifractal spectra of μ and the multifractal spectra of ν follows easily from this (see Proposition 4.47), namely,

$$\begin{aligned} f_{\mathrm{H},\mu}(\alpha) &= \max\left(\dim_{\mathrm{H}}\Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\,,\,f_{\mathrm{H},\nu}(\alpha)\right),\\ f_{\mathrm{P},\mu}(\alpha) &= \max\left(\dim_{\mathrm{P}}\Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\,,\,f_{\mathrm{P},\nu}(\alpha)\right).\end{aligned}\tag{4.45}$$

Hence, in order to prove Theorem 4.28 it suffices to compute the Hausdorff and packing dimensions of the set $\{x\in K_\varnothing\mid\dim_{\mathrm{loc}}(x;\mu)=\alpha\}$. This is done in Part 3 and Part 4 of the proof, respectively.

Part 3: Section 4.3.5. In Section 4.3.5 we obtain a lower bound for the Hausdorff dimension of the set $\{x \in K_\varnothing \mid \dim_{\mathrm{loc}}(x;\mu) = \alpha\}$. Namely, we prove (see Proposition 4.51) that

$$\beta^*(\alpha) \le \dim_{\mathrm{H}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\}. \tag{4.46}$$

Part 4: Section 4.3.6. In Section 4.3.6 we obtain an upper bound for the packing dimension of the set $\{x \in K_\varnothing \mid \dim_{\mathrm{loc}}(x;\mu) = \alpha\}$. Namely, we prove (see Proposition 4.53) that

$$\dim_{\mathrm{P}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \le \beta^*(\alpha). \tag{4.47}$$

Finally, combining (4.45), (4.46) and (4.47) gives the desired result This completes the brief overview of the proof of Theorem 4.28.

Remark. Collapse of the multifractal spectra of inhomogeneous self-similar measures. The following rather surprising result follows from Theorem 4.28. Namely, regardless of how the maps $(S_1,\ldots,S_N)$ are chosen and regardless of how the measure ν is chosen, the multifractal spectra $f_{\mathrm{H},\mu}$ and $f_{\mathrm{P},\mu}$ of μ always collapse and become identical to those of ν for all sufficiently small α and for all sufficiently large α. This is the contents of the next corollary.

Corollary 4.29. *Assume that the IOSC is satisfied. If* $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, *then*

$$\begin{aligned} f_{\mathrm{H},\mu} &= f_{\mathrm{H},\nu}\,, \\ f_{\mathrm{P},\mu} &= f_{\mathrm{P},\nu}\,. \end{aligned}$$

Proof. This follows from Theorem 4.28, since $\beta^*(\alpha) = -\infty$ for all $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$ (see Lemma 4.50). □

Remark. Inhomogeneous self-similar measures may have non-concave multifractal spectra. If follows from Theorem 4.28 that the multifractal spectra of an inhomogeneous self-similar measure may be non-concave. This is in sharp contrast to the behaviour of the multifractal spectra of (ordinary) self-similar measures satisfying the Open Set Condition. The reader is referred to Example 4.30 in Section 4.3.2 for an example of an inhomogeneous self-similar measure with highly non-concave multifractal spectra.

Remark. Inhomogeneous self-similar measures may fail the Multifractal Formalism. As we have mentioned in Section 4.1.3 Multifractal Formalism states that (1) the L^q spectra coincide, and (2) that the multifractal spectra functions coincide with the Legendre transform of the L^q spectra. Since any Legendre transform is concave, it follows that the multifractal spectra of measures satisfying the Multifractal Formalism must be concave. It therefore follows that in-homogenoues self-similar measures may fail the Multifractal Formalism. As in the previous remark, this is in sharp contrast to the behaviour of the multifractal spectra of (ordinary) self-similar measures satisfying the Open Set Condition.

4.3.2 Examples and applications

In this section we consider various applications of our main results. Many of our applications are related to the notoriously difficult problem of computing for the multifractal spectra of of self-similar measures not satisfying the OSC. We show that our results provide a systematic approach to obtain non-trivial bounds (and in some cases even exact values) for the multifractal spectra of several large and interesting classes of self-similar measures not satisfying the Open Set Condition. We will now describe the key idea in our approach. Namely, let $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ be contracting similarities (not necessarily satisfying the OSC) and let $(p_1, \ldots, p_N)$ be a probability vector. Also, let μ be the self-similar measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N)$, i.e. μ is the unique measure such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1} .$$

Iterating this identity shows that $\mu = \sum_{\mathbf{i} \in \{1,\ldots N\}^n} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1}$ for all positive integers n. Hence, if we fix $I \subseteq \{1, \ldots N\}^n$, and let

$$\begin{aligned} p &= 1 - \sum_{\mathbf{i} \in I} p_{\mathbf{i}} , \\ \nu &= \frac{1}{p} \sum_{\mathbf{i} \in \{1,\ldots N\}^n \setminus I} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1} , \end{aligned}$$

then clearly

$$\mu = \sum_{\mathbf{i} \in I} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1} + p\nu ,$$

i.e. μ is the inhomogeneous self-similar measure associated with the list $\big((S_{\mathbf{i}})_{\mathbf{i} \in I} , (p_{\mathbf{i}})_{\mathbf{i} \in I} , p, \nu \big)$. Therefore, if it is possible to choose I such that the list $\big((S_{\mathbf{i}})_{\mathbf{i} \in I} , (p_{\mathbf{i}})_{\mathbf{i} \in I} , p, \nu \big)$ satisfies the IOSC and the spectra of ν can be computed (or bounds for the spectra of ν can be obtained), then the spectra of μ can be found (or bounds for the spectra of μ can be obtained) using Theorem 4.28. Below we give several examples using this technique for finding the spectra of self-similar measures not satisfying the OSC, including, for example, self-similar measures supported on the so-called $(0, 1, 3)$-set of γ-expansions with deleted digits (see Example 4.37) and non-linear self-similar measures introduced by Glickenstein & Strichartz [GS96] (see Example 4.38).

Example 4.30. Testud measures: a class of self-similar measures not satisfying the Open Set Condition.

In this section we study the connection between inhomogeneous self-similar measures satisfying the IOSC and a class of self-similar measures introduced and inverstigated by Testud in [Tes05, Tes06]. The measures in [Tes05, Tes06] do not satisfy the OSC and the usual techniques for computing the multifractal spectra developed in [AP96, CM92] can therefore not by applied. However, despite this Testud [Tes05, Tes06] found formulas for the multifractal spectra of these measures, see Theorem 4.31 below. As an application of our results we will now obtain a simple proof of Testud's result.
We begin by describing the measures considered by Testud. For a positive integer $l \geq 2$ define functions $S_i : [0, 1] \to [0, 1]$ for $i = 0, \ldots, 2l - 1$ by $S_i(x) = \frac{1}{l}x + \frac{i}{l}$ and $S_{i+l}(x) = -\frac{1}{l}x + \frac{i+1}{l}$ for $i = 0, \ldots, l - 1$. For a probability vector $(p_0, \ldots, p_{2l-1})$ with $p_i \neq 0$ for all $i = 0, \ldots, l - 1$, Testud considered the self-similar measure μ associated with the list $(S_0, \ldots, S_{2l-1}, p_0, \ldots p_{2l-1})$, i.e. μ is the unique measure such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1} . \tag{4.48}$$

It is clear that the list $(S_0, \ldots, S_{2l-1})$ does not satisfy the OSC. Infact, many of the overlaps $S_i([0, 1]) \cap S_j([0, 1])$ are "very big". For example, if $i = 0, \ldots, l-1$ and $p_{i+l} \neq 0$, then $S_i([0, 1]) \cap S_{i+l}([0, 1]) = S_i([0, 1])$. In [Tes05, Tes06] Testud obtained the following formulas for the multifractal spectra of μ.

Theorem 4.31. *Let μ be the self-similar measure satisfying (4.48). Define the function $T:[0,1]\to[0,1]$ and the probability measure λ by*

$$\begin{aligned} T(x) &= 1-x\,,\\ \lambda &= \tfrac{\mu+\mu\circ T}{2}\,. \end{aligned}$$

Also, let $B=\{0\le i\le l-1\,|\,p_{i+l}=0\}$ and define the function $\tau:\mathbb{R}\to\mathbb{R}$ by $\tau(q)=\frac{\log\sum_{i\in B}p_i^q}{\log l}$. Then

$$\begin{aligned} f_{\mathrm{H},\mu}(\alpha) &= \max\Big(\tau^*(\alpha)\,,\,f_{\mathrm{H},\lambda}(\alpha)\Big)\,,\\ f_{\mathrm{P},\mu}(\alpha) &= \max\Big(\tau^*(\alpha)\,,\,f_{\mathrm{P},\lambda}(\alpha)\Big)\,, \end{aligned}$$

for $\alpha\ge 0$.

Remark. In fact, Testud proved Theorem 4.31 under the additional assumption that $B\cap(l-1-B)=\varnothing$. However, our proof shows that this assumption is not needed.

Remark. Theorem 4.31 provides a formula for the multifractal spectra of μ in terms of the multifractal spectra of $\lambda=\frac{\mu+\mu\circ T}{2}$ (and the function τ). The usefulness of this formula is due to the fact that frequently λ is a self-similar measure satisfying the OSC, and the multifractal spectra of λ can therefore be computed explicitely. We will elaborate on this after having proved Theorem 4.31.

We now show that μ may be viewed as an inhomogeneous self-similar measure and using this fact (together with Theorem 4.28) we give a simple proof of Theorem 4.31. We first show that μ may be viewed as an inhomogeneous self-similar measure as follows. Namely, define $p\in(0,1)$ and the probability measure ν by

$$\begin{aligned} p &= 1-\sum_{i\in B}p_i\,,\\ \nu &= \frac{1}{p}\sum_{i\in\{0,\dots,l-1\}\setminus B}\big(p_i\mu+p_{i+l}\mu\circ T\big)\circ S_i^{-1}\,. \end{aligned}$$

Then clearly

$$\mu=\sum_{i\in B}p_i\mu\circ S_i^{-1}+p\nu\,,$$

i.e. μ is the inhomogeneous self-similar measure associated with the list $\big((S_i)_{i\in B}\,,\,(p_i)_{i\in B}\,,p,\nu\big)$. Using the fact that μ is an inhomogeneous self-similar measure we will now give a simple proof of Theorem 4.31. We start by proving four small auxiliary lemmas.

Lemma 4.32. *Let $U=(0,1)$. Then $\mu(\partial U)=0$.*

Proof. Since $\partial U=\{0,1\}$, it suffices to show that $\mu(\{0\})=0$ and that $\mu(\{1\})=0$. We first show that $\mu(\{0\})=0$. Fix a positive integer n and note that $\mu(\{0\})\le\mu([0,\frac{1}{n}))=\sum_{\mathbf{i}\in\{0,\dots,2l-1\}^n}p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}[0,\frac{1}{n}))=\sum_{\mathbf{i}\in\{0,l\}^n}p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}[0,\frac{1}{n}))\le\sum_{\mathbf{i}\in\{0,l\}^n}p_{\mathbf{i}}=(p_0+p_l)^n$. We conclude from this that $\mu(\{0\})\le\lim_n(p_0+p_l)^n=0$. Similarly we can prove that $\mu(\{1\})=0$. $\square$

Lemma 4.33. *The list* $\big((S_i)_{i\in B}\,,\,(p_i)_{i\in B}\,,p,\nu\big)$ *satisfies the IOSC with* $U=(0,1)$.

Proof. It is clear that Conditions 1–3 are satisfied. Next, we show that Conditions 4 and 5 are satisfied. Observe that $C=\operatorname{supp}\nu=\cup_{i\in\{0,\dots,l-1\}\setminus B}\overline{S_iU}$.
We first prove that if $i\in B$, then

$$\dim_{\mathrm{H}}(\overline{S_iU}\cap C)=\dim_{\mathrm{P}}(\overline{S_iU}\cap C)=0\,,\quad \dim_{\mathrm{H}}(\partial U\cap C)=\dim_{\mathrm{P}}(\partial U\cap C)=0\,. \tag{4.49}$$

Indeed, note that if $i\in B$, then

$$\overline{S_iU}\cap C\subseteq\partial S_iU\,,\quad \partial U\cap C\subseteq\partial U\,. \tag{4.50}$$

Since ∂S_iU and ∂U are finite sets (in fact, each set consists of two elements), the equalities in (4.49) follow immediately from (4.50). This proves (4.49).
Next we prove if $i\in B$, then

$$\nu(\overline{S_iU}\cap C)=0\,,\quad \nu(\partial U\cap C)=0\,. \tag{4.51}$$

It follows from (4.50) that if $j\in\{0,\dots,2l-1\}$, then

$$S_j^{-1}(\overline{S_iU}\cap C)\subseteq S_j^{-1}(\partial S_iU)\subseteq\partial U\,,\quad S_j^{-1}(\partial U\cap C)\subseteq S_j^{-1}(\partial U)\subseteq\partial U\,. \tag{4.52}$$

Finally, since $T\partial U=\partial U$, we now conclude from (4.52) and Lemma 4.32 that if $E=\overline{S_iU}\cap C$ or if $E=\partial U\cap C$, then

$$\begin{aligned}
\nu(E) &= \frac{1}{p}\sum_{j\in\{0,\dots,l-1\}\setminus B}\Big(p_j\mu(S_j^{-1}E)+p_{j+l}\mu(T(S_j^{-1}E))\Big)\\
&\le \frac{1}{p}\sum_{j\in\{0,\dots,l-1\}\setminus B}\Big(p_j\mu(\partial U)+p_{j+l}\mu(T\partial U)\Big)\\
&= \frac{1}{p}\sum_{j\in\{0,\dots,l-1\}\setminus B}\Big(p_j\mu(\partial U)+p_{j+l}\mu(\partial U)\Big)\\
&= 0\,.
\end{aligned}$$

This proves (4.51) and completes the proof of Lemma 4.33. □

Lemma 4.34. *We have*

$$\begin{aligned}
f_{\mathrm{H},\nu}(\alpha) &= f_{\mathrm{H},\Sigma_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha)\,,\\
f_{\mathrm{P},\nu}(\alpha) &= f_{\mathrm{P},\Sigma_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha)\,,
\end{aligned}$$

for all $\alpha\ge 0$.

Proof. It is easily seen that,

$$\frac{2p_{\min}}{p}\sum_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}\le\nu\le\frac{2p_{\max}}{p}\sum_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}\,,$$

where $p_{\min}=\min\big(\min_{i\in\{0,\dots,l-1\}\setminus B}p_i\,,\,\min_{i\in\{0,\dots,l-1\}\setminus B}p_{i+l}\big)$ and $p_{\max}=\max\big(\max_{i\in\{0,\dots,l-1\}\setminus B}p_i\,,\,\max_{i\in\{0,\dots,l-1\}\setminus B}p_{i+l}\big)$. Namely, there are constants $c_{\min}=\frac{2p_{\min}}{p},c_{\max}=\frac{2p_{\max}}{p}>0$ such that $c_{\min}\sum_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}\le\nu\le c_{\max}\sum_{i\in\{0,\dots,l-1\}\setminus B}\lambda\circ S_i^{-1}$.
The desired result follows immediately from this inequality. □

Lemma 4.35. *We have*

$$\begin{aligned} f_{\mathrm{H},\Sigma_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha) &= f_{\mathrm{H},\lambda}(\alpha)\,, \\ f_{\mathrm{P},\Sigma_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha) &= f_{\mathrm{P},\lambda}(\alpha)\,, \end{aligned}$$

for all $\alpha \geq 0$ *.*

Proof. We clearly have

$$\begin{aligned} & f_{\mathrm{H},\Sigma_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha) \\ &\quad = \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\sum_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\} \\ &\quad = \dim_{\mathrm{H}}\left\{x \in \bigcup_{i\in\{0,\ldots,l-1\}\setminus B}\operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\} \\ &\quad = \dim_{\mathrm{H}}\bigcup_{i\in\{0,\ldots,l-1\}\setminus B}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\} \\ &\quad = \max_{i\in\{0,\ldots,l-1\}\setminus B}\dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\}. \end{aligned} \tag{4.53}$$

However, since the set $F = \cup_{i,j\in\{0,\ldots,l-1\}\setminus B\,,\,i\neq j}\big(\operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \cap \operatorname{supp}\left(\lambda\circ S_j^{-1}\right)\big)$ is finite, it follows that if $i \in \{0,\ldots,l-1\}\setminus B$, then

$$\begin{aligned} & \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\} \\ &\quad = \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right)\setminus F \,\middle|\, \dim_{\mathrm{loc}}\left(x;\sum_{j\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_j^{-1}\right)=\alpha\right\} \\ &\quad = \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right)\setminus F \,\middle|\, \dim_{\mathrm{loc}}\left(x;\lambda\circ S_i^{-1}\right)=\alpha\right\} \\ &\quad = \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}\left(\lambda\circ S_i^{-1}\right) \,\middle|\, \dim_{\mathrm{loc}}\left(x;\lambda\circ S_i^{-1}\right)=\alpha\right\} \\ &\quad = f_{\mathrm{H},\lambda\circ S_i^{-1}}(\alpha)\,. \end{aligned} \tag{4.54}$$

Combining (4.53) and (4.54) we now conclude that

$$f_{\mathrm{H},\Sigma_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha) = \max_{i\in\{0,\ldots,l-1\}\setminus B} f_{\mathrm{H},\lambda\circ S_i^{-1}}(\alpha)\,. \tag{4.55}$$

Next, since S_i is a similarity it is easily seen that $f_{\mathrm{H},\lambda\circ S_i^{-1}}(\alpha) = f_{\mathrm{H},\lambda}(\alpha)$ for all i (see, for example, [[Fal97], Exercise 11.9]), whence

$$\max_{i\in\{0,\ldots,l-1\}\setminus B} f_{\mathrm{H},\lambda\circ S_i^{-1}}(\alpha) = f_{\mathrm{H},\lambda}(\alpha)\,. \tag{4.56}$$

Finally, combining (4.55) and (4.56) gives the desired result. The proof of the formula for the packing spectrum is very similar. This completes the proof. □

We can now prove Theorem 4.31.

Proof of Theorem 4.31
It follows from Lemma 4.33 that the list $\big((S_i)_{i\in B}\,,\,(p_i)_{i\in B}\,,p,\nu\big)$ satisfies the IOSC, and we therefore deduce from Theorem 4.28 that

$$f_{\mathrm{H},\mu}(\alpha)=\max\Big(\tau^*(\alpha)\,,\,f_{\mathrm{H},\nu}(\alpha)\Big)\,, \tag{4.57}$$

for $\alpha\geq 0$, where $\tau:\mathbb{R}\to\mathbb{R}$ is defined by $\sum_{i\in B}p_i^q(\frac{1}{l})^{\tau(q)}=1$, i.e. $\tau(q)=\frac{\log\sum_{i\in B}p_i^q}{\log l}$. Also, Lemma 4.34 and Lemma 4.35 imply that

$$f_{\mathrm{H},\nu}(\alpha)=f_{\mathrm{H},\Sigma_{i\in\{0,\ldots,l-1\}\setminus B}\lambda\circ S_i^{-1}}(\alpha)=f_{\mathrm{H},\lambda}(\alpha) \tag{4.58}$$

for $\alpha\geq 0$. The desired result follows from (4.57) and (4.58). The formula for $f_{\mathrm{P},\mu}(\alpha)$ is proved similarly.

We now consider a concrete example of Theorem 4.31. Let $l=3$. Fix two positive real numbers s and t with $3s+2t\leq 1\leq 3s+3t$ (for example, we may take $s=\frac{1}{12}$ and $t=\frac{1}{3}$), and define the probability vector $(p_i)_{i=0,\ldots,5}$ by

$$p_0=s+t\,,\quad p_1=1-2(s+t)\,,\quad p_2=s\,,\quad p_3=t\,,\quad p_4=p_5=0\,.$$

In this case $B=\{1,2\}$ and $\tau(q)=\frac{\log(p_1^q+p_2^q)}{\log 3}$. It follows from Theorem 4.31 that

$$f_{\mathrm{H},\mu}(\alpha)=\max\Big(\tau^*(\alpha)\,,\,f_{\mathrm{H},\lambda}(\alpha)\Big)\,, \tag{4.59}$$

$$f_{\mathrm{P},\mu}(\alpha)=\max\Big(\tau^*(\alpha)\,,\,f_{\mathrm{P},\lambda}(\alpha)\Big)\,, \tag{4.60}$$

for $\alpha\geq 0$.
We will now prove that $\lambda=\frac{\mu+\mu\circ T}{2}$ is, in fact, a self-similar measure. More precisely we will now prove the following claim. We note similar calculations also appear in [Tes05, Tes06]. However, we have decided to include the simple and very brief calculation in Claim 4.36 for completeness.

Claim 4.36. *We have*

$$\lambda=p_0\lambda\circ S_0^{-1}+p_1\lambda\circ S_1^{-1}+(p_2+p_3)\lambda\circ S_2^{-1}\,,$$

i.e. λ *is the self-similar measure associated with the list* $(S_0,S_1,S_2,p_0,p_1,p_2+p_3)$. *In particular, since the list* (S_0,S_1,S_2) *clearly satisfies the OSC, we conclude (see [Fal97]) that if* $b:\mathbb{R}\to\mathbb{R}$ *is defined by* $b(q)=\frac{\log(p_0^q+p_1^q+(p_2+p_3)^q)}{\log 3}$, *then* $f_{\mathrm{H},\lambda}(\alpha)=f_{\mathrm{P},\lambda}(\alpha)=b^*(\alpha)$ *for* $\alpha\geq 0$.

Proof. First observe that

$$S_3^{-1}=T\circ S_0^{-1}\,,\quad S_0^{-1}\circ T=T\circ S_2^{-1}\,,\quad S_1^{-1}\circ T=T\circ S_1^{-1}\,,\quad S_2^{-1}\circ T=T\circ S_0^{-1}\,,\quad S_3^{-1}\circ T=S_2^{-1}\,.$$

It follows from this and the definition of μ that

$$\begin{aligned}\mu&=p_0\mu\circ S_0^{-1}+p_1\mu\circ S_1^{-1}+p_2\mu\circ S_2^{-1}+p_3\mu\circ S_3^{-1}\\&=p_0\mu\circ S_0^{-1}+p_1\mu\circ S_1^{-1}+p_2\mu\circ S_2^{-1}+p_3\mu\circ T\circ S_0^{-1}\,,\\\mu\circ T&=\big(p_0\mu\circ S_0^{-1}+p_1\mu\circ S_1^{-1}+p_2\mu\circ S_2^{-1}+p_3\mu\circ S_3^{-1}\big)\circ T\end{aligned} \tag{4.61}$$

$$\begin{aligned}
&= p_0\mu\circ S_0^{-1}\circ T + p_1\mu\circ S_1^{-1}\circ T + p_2\mu\circ S_2^{-1}\circ T + p_3\mu\circ S_3^{-1}\circ T\\
&= p_0\mu\circ T\circ S_2^{-1} + p_1\mu\circ T\circ S_1^{-1} + p_2\mu\circ T\circ S_0^{-1} + p_3\mu\circ S_2^{-1}\,. \qquad (4.62)
\end{aligned}$$

Adding (4.61) and (4.62) and using the fact that $p_0 = p_2 + p_3$ gives

$$\begin{aligned}
\tfrac{\mu+\mu\circ T}{2} &= \tfrac{p_0\mu+(p_2+p_3)\mu\circ T}{2}\circ S_0^{-1} + p_1\tfrac{\mu+\mu\circ T}{2}\circ S_1^{-1} + \tfrac{(p_2+p_3)\mu+p_0\mu\circ T}{2}\circ S_2^{-1}\\
&= p_0\tfrac{\mu+\mu\circ T}{2}\circ S_0^{-1} + p_1\tfrac{\mu+\mu\circ T}{2}\circ S_1^{-1} + (p_2+p_3)\tfrac{\mu+\mu\circ T}{2}\circ S_2^{-1}\,.
\end{aligned}$$

This completes the proof of Claim 4.36. □

Finally, combining (4.59) and Claim 4.36, we conclude that if we define $\tau, b : \mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}
\tau(q) &= \tfrac{\log(p_1^q+p_2^q)}{\log 3} = \tfrac{\log((1-2(s+t))^q+s^q)}{\log 3}\,,\\
b(q) &= \tfrac{\log(p_0^q+p_1^q+(p_2+p_3)^q)}{\log 3} = \tfrac{\log(2(s+t)^q+(1-2(s+t))^q)}{\log 3}\,,
\end{aligned}$$

then

$$f_{\mathrm{H},\mu}(\alpha) = f_{\mathrm{P},\mu}(\alpha) = \max\Big(\tau^*(\alpha)\,,\, b^*(\alpha)\Big)\,,$$

for $\alpha \geq 0$. It follows from standard properties of the Legendre transform that

$$\begin{aligned}
\tau^*(\alpha) &\geq 0 &&\text{for } \alpha\in\Big[\tfrac{\log\max(p_1,p_2)}{-\log 3}, \tfrac{\log\min(p_1,p_2)}{-\log 3}\Big] = \Big[\tfrac{\log(1-2(s+t))}{-\log 3}, \tfrac{\log s}{-\log 3}\Big]\,,\\
\tau^*(\alpha) &= -\infty &&\text{for } \alpha\notin\Big[\tfrac{\log\max(p_1,p_2)}{-\log 3}, \tfrac{\log\min(p_1,p_2)}{-\log 3}\Big] = \Big[\tfrac{\log(1-2(s+t))}{-\log 3}, \tfrac{\log s}{-\log 3}\Big]\,,
\end{aligned}$$

and

$$\begin{aligned}
b^*(\alpha) &\geq 0 &&\text{for } \alpha\in\Big[\tfrac{\log\max(p_0,p_1,p_2+p_3)}{-\log 3}, \tfrac{\log\min(p_0,p_1,p_2+p_3)}{-\log 3}\Big] = \Big[\tfrac{\log(s+t)}{-\log 3}, \tfrac{\log(1-2(s+t))}{-\log 3}\Big]\,,\\
b^*(\alpha) &= -\infty &&\text{for } \alpha\notin\Big[\tfrac{\log\max(p_0,p_1,p_2+p_3)}{-\log 3}, \tfrac{\log\min(p_0,p_1,p_2+p_3)}{-\log 3}\Big] = \Big[\tfrac{\log(s+t)}{-\log 3}, \tfrac{\log(1-2(s+t))}{-\log 3}\Big]\,.
\end{aligned}$$

We observe that the multifractal spectra $f_{\mathrm{H},\mu}(\alpha)$ and $f_{\mathrm{P},\mu}(\alpha)$ are highly non-concave. In fact, if we write

$$\Delta(\alpha) = \Big\{x\in\mathbb{R}\,\Big|\, \dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\,,$$

then the set

$$\Big\{\alpha\in\mathbb{R}\,\Big|\, \dim_{\mathrm{H}}\Delta(\alpha)>0\Big\}$$

consists of two disjoint intervals I and J, namely $I = \big(\tfrac{\log(s+t)}{-\log 3}, \tfrac{\log(1-2(s+t))}{-\log 3}\big)$ and $J = \big(\tfrac{\log(1-2(s+t))}{-\log 3}, \tfrac{\log s}{-\log 3}\big)$, such that $f_{\mathrm{H},\mu}$ is strictly concave on both I and J. In Figure 4.3.1 we sketch the graphs of τ^* and b^* for $s=\frac{1}{12}$ and $t=\frac{1}{3}$.

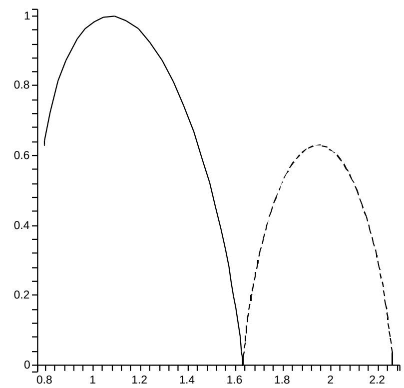

Figure 4.3.1:

The graph of the multifractal spectra

$$f_{\mathrm{H},\mu}(\alpha) = f_{\mathrm{P},\mu}(\alpha) = \max(\,\tau^*(\alpha)\,,\,b^*(\alpha)\,)$$

of the self-similar measure μ in Example 4.30 for $s = \frac{1}{12}$ and $t = \frac{1}{3}$. The dashed line represents the graph of the function τ^* and the solid line represents the graph of the function b^*.

Example 4.37. Self-similar measures supported by the $(0,1,3)$-set of γ-expansions with deleted digits.

We now present another example of our main results. Fix $\gamma \in (\frac{1}{4}, \frac{1}{3})$. Define $S_0, S_1, S_3 : \mathbb{R} \to \mathbb{R}$ by $S_0(x) = \gamma x$, $S_1(x) = \gamma x + \gamma$ and $S_3(x) = \gamma x + 3\gamma$, and let (p_0, p_1, p_3) be a probability vector. Let μ_γ denote the self-similar measure associated with the list $(S_0, S_1, S_3, p_0, p_1, p_3)$, i.e. μ_γ is the unique measure such that

$$\mu_\gamma = \sum_i p_i \mu_\gamma \circ S_i^{-1}\,. \tag{4.63}$$

The main difficulty in analyzing the multifractal spectrum of μ_γ for $\gamma \in (\frac{1}{4}, \frac{1}{3})$ is due to the fact that the OSC is not satisfied and the standard results developed in [AP96, CM92] can therefore not be applied. We will now show the results and techniques developed in this paper provides a non-trivial lower bound for the multifractal spectra of μ_γ.

The iterated function system (S_0, S_1, S_3) has recently attracted considerable interest due to its relationship with the so-called $(0,1,3)$-set of γ-expansions with deleted digits. Motivated by problems of Palis & Takens [PT93] on arithmetic sums of Cantor sets, Keane (see [KSS95]) asked the following question. Namely, for $\gamma \in (0,1)$ he considered the set Γ_γ of numbers whose γ-expansion only contains the digits $0, 1$ and 3, i.e.

$$\Gamma_\gamma = \left\{ \sum_{n=1}^{\infty} a_n \gamma^n \;\middle|\; a_n \in \{0,1,3\} \text{ for } n \in \mathbb{N} \right\},$$

and asked whether the Hausdorff dimension, $\dim_{\mathrm{H}} \Gamma_\gamma$, of Γ_γ is a continuous function of γ for $\gamma \in (\frac{1}{4}, \frac{1}{3})$. Since clearly $\Gamma_\gamma = \cup_i S_i(\Gamma_\gamma)$, we conclude that Γ_γ is the self-similar set associated with the list (S_0, S_1, S_3). However, the main difficulty in analyzing the Hausdorff dimension of Γ_γ for $\gamma \in (\frac{1}{4}, \frac{1}{3})$ is due to the fact that Γ_γ is a self-similar set which does not satisfy the OSC and the standard results developed by Hutchinson [Hut81] can therefore not be applied. Pollicott & Simon gave the negative answer to Keane's question in [PS95] (see also [KSS95]). Finally, since Γ_γ is the self-similar set associated with the list (S_0, S_1, S_3) it follows that

$$\operatorname{supp}\mu_\gamma = \Gamma_\gamma\,,$$

i.e. μ_γ is supported on the $(0,1,3)$-set of γ-expansions with deleted digits.
Iterating (4.63) we see that $\mu_\gamma = \sum_{\mathbf{i}\in\{0,1,3\}^3} p_{\mathbf{i}}\mu_\gamma\circ S_{\mathbf{i}}^{-1}$. In particular, if we write

$$I = \big\{\,013\,,\,113\,,\,133\,,\,313\,,\,333\,\big\} \subseteq \{0,1,3\}^3\,,$$

and let

$$\begin{aligned} p &= 1-\sum_{\mathbf{i}\in I} p_{\mathbf{i}}\,, && (4.64)\\ \nu &= \frac{1}{p}\sum_{\mathbf{i}\in\{0,1,3\}^3\setminus I} p_{\mathbf{i}}\mu_\gamma\circ S_{\mathbf{i}}^{-1}\,, && (4.65)\end{aligned}$$

then clearly

$$\mu_\gamma = \sum_{\mathbf{i}\in I} p_{\mathbf{i}}\mu_\gamma\circ S_{\mathbf{i}}^{-1} + p\nu\,,$$

i.e. μ_γ is the inhomogeneous self-similar measure associated with the list $\big((S_{\mathbf{i}})_{\mathbf{i}\in I}\,,\,(p_{\mathbf{i}})_{\mathbf{i}\in I}\,,p,\nu\big)$. Using the fact that μ_γ is an inhomogeneous self-similar measure we will now obtain lower bounds for the multifractal spectra of μ_γ. Let $U=(0,\frac{3\gamma}{1-\gamma})$. We first show that the list $\big((S_{\mathbf{i}})_{\mathbf{i}\in I}\,,\,(p_{\mathbf{i}})_{\mathbf{i}\in I}\,,p,\nu\big)$ satisfies the IOSC with open set equal to U. Indeed, it is not difficult to see that the list $(S_{\mathbf{i}})_{\mathbf{i}\in I}$ satisfies the OSC with open set U. It is also clear that $\operatorname{supp}\mu_\gamma\subseteq\overline{U}$, whence $C=\operatorname{supp}\nu\subseteq \cup_{\mathbf{i}\in\{0,1,3\}^3\setminus I}S_{\mathbf{i}}(\overline{U})$, and it is not difficult to see (since $S_{03}(\overline{U})\cap S_{13}(\overline{U})=\varnothing$) that this implies that $\overline{S_{\mathbf{i}}U}\cap C\subseteq\overline{S_{\mathbf{i}}U}\cap(\cup_{\mathbf{j}\in\{0,1,3\}^3\setminus I}S_{\mathbf{j}}(\overline{U}))=\varnothing$ for all $\mathbf{i}\in I$ and that $\partial U\cap C\subseteq\partial U\cap(\cup_{\mathbf{j}\in\{0,1,3\}^3\setminus I}S_{\mathbf{j}}(\overline{U}))=\{0\}$. However, one can show using techniques described in Lemma 4.32 that $\nu(\{0\})=0$. It follows immediately from this that the list $\big((S_{\mathbf{i}})_{\mathbf{i}\in I}\,,\,(p_{\mathbf{i}})_{\mathbf{i}\in I}\,,p,\nu\big)$ satisfies the IOSC with open set equal to U.
Since the IOSC is satisfied we now conclude from Theorem 4.28 that if we define $\beta:\mathbb{R}\to\mathbb{R}$ by

$$\beta(q) = \tfrac{\log(\sum_{\mathbf{i}\in I}p_{\mathbf{i}}^q)}{-\log\gamma^3} = \tfrac{\log((p_0p_1p_3)^q+(p_1^2p_3)^q+2(p_1p_3^2)^q+(p_3^3)^q)}{-3\log\gamma}\,,$$

then

$$\begin{aligned} f_{\mathrm{H},\mu_\gamma}(\alpha) &= \max\Big(\beta^*(\alpha)\,,\,f_{\mathrm{H},\nu}(\alpha)\Big)\geq\beta^*(\alpha)\,,\\ f_{\mathrm{P},\mu_\gamma}(\alpha) &= \max\Big(\beta^*(\alpha)\,,\,f_{\mathrm{P},\nu}(\alpha)\Big)\geq\beta^*(\alpha)\,,\end{aligned}$$

for $\alpha\geq 0$. This provides non-trivial lower bounds for the multifractal spectra of μ.

Example 4.38. Non-linear self-similar measures.

We consider probability measures μ on $\mathbb{R}^d$ satisfying the following non-linear self-similar identity

$$\mu = \sum_{i=1}^N p_i\mu\circ S_i^{-1} + \sum_{j=1}^M q_j(\mu*\mu)\circ T_j^{-1}\,, \tag{4.66}$$

where $(p_1,\dots,p_N,q_1,\dots,q_M)$ is a probability vector and $S_i,T_j:\mathbb{R}^d\to\mathbb{R}^d$ are contracting similarities and the contracting ratios of the T_j 's are less than $\frac{1}{2}$; the existence and uniqueness of measures μ satisfying (4.66) is proved in [GS96] who also analyzed the asymptotic behaviour of the Fourier transform of μ (for more details on the asymptotic behaviour of the Fourier transform of μ see Section 5.2.3). We note that measures μ satisfying the non-linear self-similar identity in (4.66) can be viewed as inhomogeneous self-similar measure as follows. Namely, define $p\in(0,1)$ and the probability measure ν by

$$\begin{aligned} p &= 1-\sum_i p_i\,, \\ \nu &= \frac{1}{p}\sum_j q_j(\mu*\mu)\circ T_j^{-1}\,. \end{aligned}$$

Then clearly

$$\mu=\sum_i p_i\mu\circ S_i^{-1}+p\nu\,,$$

i.e. μ is the inhomogeneous self-similar measure associated with the list $(S_1,\dots,S_N,p_1,\dots,p_N,p,\nu)$. Using the fact that μ is an inhomogeneous self-similar measure we will now give non-trivial lower bounds for the multifractal spectra of μ. Indeed, the following result follows from Theorem 4.28.

Theorem 4.39. *Let μ be a non-linear self-similar measure satisfying (4.66). Assume that there is a non-empty, bounded and open set U satisfying*
(a) For all $i=1,\dots,N$, we have $S_iU\subseteq U$;
(b) For all $i,j=1,\dots,N$ with $i\neq j$, we have $S_iU\cap S_jU=\varnothing$;
(c) $U\cap K_\varnothing\neq\varnothing$ where $K_\varnothing$ is the non-empty compact set such that $K_\varnothing=\cup_iS_iK_\varnothing$;
(d) For all $j=1,\dots,M$, we have $T_j(\overline{U}+\overline{U})\subseteq U$;
(e) For all $i=1,\dots,N$, we have $S_i\overline{U}\cap(\cup_jT_j(\overline{U}+\overline{U}))=\varnothing$.
(Conditions (a)–(e) ensure that the IOSC is satisfied. Indeed, Conditions 1–3 of the IOSC will follow from (a)–(c); Conditions 4 and 6 of the IOSC will follow from (d); and Condition 5 of the IOSC will follow from (e).) Let r_i denote the similarity ratio of S_i and define $\beta:\mathbb{R}\to\mathbb{R}$ by

$$\sum_i p_i^q r_i^{\beta(q)}=1\,.$$

Then

$$f_{\mathrm{P},\mu}(\alpha)\geq f_{\mathrm{H},\mu}(\alpha)\geq\beta^*(\alpha)\,,$$

for $\alpha\geq0$.

Proof. The conclusion clearly follows from Theorem 4.28 provided the list $(S_1,\dots,S_N,p_1,\dots,p_N,p,\nu)$ satisfies the IOSC. Below we prove that the list $(S_1,\dots,S_N,p_1,\dots,p_N,p,\nu)$ satisfies the IOSC. Conditions 1–3 of the IOSC follow immidiately from (a)–(c). We will now prove that Conditions (4)–(6) of the IOSC are satisfied. As usual, we write K_C for the support of μ and we write C for the support of ν. Note that

$$C=\operatorname{supp}\nu=\bigcup_j T_j(\operatorname{supp}(\mu*\mu))=\bigcup_j T_j(\operatorname{supp}\mu+\operatorname{supp}\mu)=\bigcup_j T_j(K_C+K_C)\,. \tag{4.67}$$

Next, observe that in order to prove Conditions (4)–(6) of the IOSC it suffices to show that $C\subseteq U$ and $S_i\overline{U}\cap C=\varnothing$ for all i.
We first prove that

$$K_C\subseteq\overline{U}\,. \tag{4.68}$$

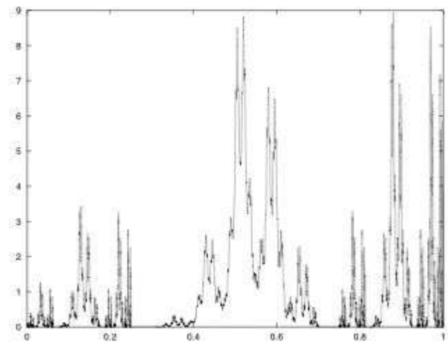

Figure 4.3.2:
The measure μ satisfying the non-linear self-similar identity

$$\mu = \sum_{i=1}^{2} p_i \mu \circ S_i^{-1} + q_1 (\mu * \mu) \circ T_1^{-1} ,$$

where the maps $S_1, S_2, T_1 : \mathbb{R} \to \mathbb{R}$ are defined by $S_1(x) = \frac{1}{4}x$, $S_2(x) = \frac{1}{4}x + \frac{3}{4}$ and $T_1(x) = \frac{1}{5}x + \frac{3}{10}$, and $(p_1, p_2, q_1) = (0.1, 0.26, 0.64)$.

In order to prove (4.68), let $\mathtt{K}(\mathbb{R}^d)$ denote the family of non-empty and compact subsets of $\mathbb{R}^d$ and equip $\mathtt{K}(\mathbb{R}^d)$ with the Hausdorff metric. Define $\mathcal{T} : \mathtt{K}(\mathbb{R}^d) \to \mathtt{K}(\mathbb{R}^d)$ by $\mathcal{T}(A) = \cup_i S_i(A) \cup \cup_j T_j(A + A)$. Using this notation, $K_C = \operatorname{supp} \mu$ is the unique element $K_C \in \mathtt{K}(\mathbb{R}^d)$ such that $K_C = \mathcal{T}(K)$, and it therefore follows from Banach's fixed point theorem that $\mathcal{T}^n(\overline{U}) \to K_C$ with respect to the Hausdorff metric. However, Conditions (a) and (d) imply that $\mathcal{T}(\overline{U}) \subseteq \overline{U}$, whence $\overline{U} \supseteq \mathcal{T}(\overline{U}) \supseteq \mathcal{T}^2(\overline{U}) \supseteq \ldots$, and so $K_C = \lim_n \mathcal{T}^n(\overline{U}) = \cap_n \mathcal{T}^n(\overline{U}) \subseteq \overline{U}$. This proves (4.68).
We can now prove that $C \subseteq U$ and that $S_i\overline{U} \cap C = \varnothing$ for all i . Indeed, using (4.67) and (4.68) we conclude that

$$C = \bigcup_j T_j(K_C + K_C) \subseteq \bigcup_j T_j(\overline{U} + \overline{U}) \subseteq U ,$$

and that

$$S_i\overline{U} \cap C = S_i\overline{U} \cap \left(\bigcup_j T_j(K_C + K_C) \right) \subseteq S_i\overline{U} \cap \left(\bigcup_j T_j(\overline{U} + \overline{U}) \right) = \varnothing .$$

This completes the proof. $\square$

We now consider a concrete example of Theorem 4.39 with $N = 2$ and $M = 1$. Define $S_1, S_2, T_1 : \mathbb{R} \to \mathbb{R}$ by $S_1(x) = \frac{1}{4}x$, $S_2(x) = \frac{1}{4}x + \frac{3}{4}$ and $T_1(x) = \frac{1}{5}x + \frac{3}{10}$. For a fixed probability vector (p_1, p_2, q_1) , let μ be the non-linear self-similar measure satisfying (4.66). We note that Conditions (a)–(e) in Theorem 4.39 are satisfied with $U = (0, 1)$. Hence, if we define $\beta : \mathbb{R} \to \mathbb{R}$ by $\beta(q) = \frac{\log(p_1^q + p_2^q)}{\log 4}$, then $f_{\mathrm{P},\mu}(\alpha) \geq f_{\mathrm{H},\mu}(\alpha) \geq \beta^*(\alpha)$ for $\alpha \geq 0$. In Figure 4.3.2 we sketch the measure μ for $(p_1, p_2, q_1) = (0.1, 0.26, 0.64)$.

Example 4.40. Discrete measures with non-trivial multifractal spectra.

Using our results in this section we will construct a large class of discrete measures with non-trivial multifractal spectra. Namely, let μ be the inhomogeneous self-similar measure satisfying the identity

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + p\nu \,, \tag{4.69}$$

where $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ are contracting similarities, $(p_1, \ldots, p_N, p)$ is a probability vector and and ν is a probability measure with compact support. We now show that if the support of ν is finite and the sets $S_1(K_C), \ldots, S_N(K_C), C$ (where, as usual, we write K_C for the support of μ and we write C for the support of ν) are pairwise disjoint, then μ is a discrete measure with non-trivial multifractal spectra. We note that other discrete measures with non-trivial multifractal spectra have been constructed by Aversa & Bandt [AB90].

Theorem 4.41. *Let μ be non-linear self-similar measure satisfying (4.69) and assume that $p \neq 0$. As usual, write $K_C = \operatorname{supp} \mu$ and $C = \operatorname{supp} \nu$. Assume that the sets $(S_1(K_C), \ldots, S_N(K_C), C)$ are pairwise disjoint and that C is a finite set. Let O denote the orbital set, i.e. $O = \cup_{\mathbf{i} \in \Sigma^*} S_{\mathbf{i}} C$, cf. (2.9). Then the following holds.*

1. *The orbital set O is countable and has full measure, i.e. $\mu(O) = 1$. In particular, the measure μ is discrete.*

2. *Let r_i denote the contracting ratio of S_i and, as usual, define $\beta : \mathbb{R} \to \mathbb{R}$ by $\sum_i p_i^q r_i^{\beta(q)} = 1$. Then $f_{\mathrm{P},\mu}(\alpha) = f_{\mathrm{H},\mu}(\alpha) = \beta^*(\alpha)$ for $\alpha \geq 0$. In particular, the multifractal spectra of μ are non-trivial.*

Proof. 1. This follows from Theorem 2.12.
2. First observe that if we write $U_r = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, K_C) < r\}$ for $r > 0$, then it follows easily from the fact that the sets $S_1(K_C), \ldots, S_N(K_C), C$ are pairwise disjoint that the list $(S_1, .., .., S_N, p_1, \ldots, p_N, p, \nu)$ satisfies the IOSC with open set U_r for all sufficiently small $r > 0$. Next, since $\operatorname{supp} \nu = C$ is finite, we conclude that $f_{\mathrm{H},\nu}(\alpha) = 0$ for all α, and it therefore follows from Theorem 4.28 and the fact that the IOSC is satisfied that $f_{\mathrm{H},\mu}(\alpha) = \max(\, \beta^*(\alpha)\,,\, f_{\mathrm{H},\nu}(\alpha)\,) = \beta^*(\alpha)$ for all α. The formula for $f_{\mathrm{P},\mu}(\alpha)$ is proved very similarly. ◻

We now consider a concrete example of Theorem 4.41. Define $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ by $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$, and let $\nu = \delta_{\frac{1}{2}}$. For a fixed probability vector (p_1, p_2, p), let μ be the inhomogeneous self-similar measure satisfying (4.69). We note that if we write $K_C = \operatorname{supp} \mu$ and $C = \operatorname{supp} \nu = \{\frac{1}{2}\}$, then the sets $S_1(K_C), S_2(K_C), C$ are pairwise disjoint. Theorem 2.8 therefore shows that μ is a discrete measure supported on the countable set $\{S_{\mathbf{i}}(\frac{1}{2}) \mid \mathbf{i} \in \{1,2\}^{\mathbb{N}}\}$ and that if we define $\beta : \mathbb{R} \to \mathbb{R}$ by $\beta(q) = \frac{\log(p_1^q + p_2^q)}{\log 3}$, then $f_{\mathrm{P},\mu}(\alpha) = f_{\mathrm{H},\mu}(\alpha) = \beta^*(\alpha)$ for $\alpha \geq 0$.

4.3.3 Proof of Theorem 4.28: Preliminary results-part 1

The purpose of this section is to prove Proposition 4.44 saying that if U is the open set in the IOSC then $\mu(S_{\mathbf{i}} K_C) = \mu(\overline{S_{\mathbf{i}} U}) = p_{\mathbf{i}}$ for $\mathbf{i} \in \Sigma^*$. However, we first observe that iterating (2.4) shows that

$$\mu = \sum_{\substack{\mathbf{i} \in \Sigma^* \\ |\mathbf{i}| = n}} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1} \;+\; p \sum_{\substack{\mathbf{i} \in \Sigma^* \\ |\mathbf{i}| < n}} p_{\mathbf{i}} \nu \circ S_{\mathbf{i}}^{-1}$$

for all positive integers n. This result will be used frequently below without further mentioning. We now state and prove Lemma 4.42 and Lemma 4.43. Finally, after having proved Lemma 4.42 and Lemma 4.43 we state and prove Proposition 4.44.

Lemma 4.42. *Assume that the IOSC is satisfied and let U be the open set in the IOSC. We have*

$$K_C \subseteq \overline{U}\,.$$

Proof. Let $\mathsf{K}(\mathbb{R}^d)$ denote the family of non-empty and compact subsets of $\mathbb{R}^d$ and equip $\mathsf{K}(\mathbb{R}^d)$ with the Hausdorff metric d_h. Next, define $\mathcal{T} : \mathsf{K}(\mathbb{R}^d) \to \mathsf{K}(\mathbb{R}^d)$ by $\mathcal{T}(A) = \cup_i S_i(A) \cup C$. Since $(\mathsf{K}(\mathbb{R}^d), d_h)$ is complete and $\mathcal{T}$ is a contraction (see the proof of Proposition 2.7), it follows from Banach's fixed point theorem that if A is compact, then $\mathcal{T}^n(A) \to K_C$ with respect to the Hausdorff metric. In particular, we see that $\mathcal{T}^n(\overline{U}) \to K_C$. However, since $S_i U \subseteq U$ and $C \subseteq \overline{U}$ we deduce that $\overline{U} \supseteq \mathcal{T}(\overline{U}) \supseteq \mathcal{T}^2(\overline{U}) \supseteq \ldots$, whence $K_C = \lim_n \mathcal{T}^n(\overline{U}) \subseteq \overline{U}$. □

Lemma 4.43. *Assume that the IOSC is satisfied and let U be the open set in the IOSC.*

1. *For all $\mathbf{i},\mathbf{j} \in \Sigma^*$ with $|\mathbf{i}| = |\mathbf{j}|$ and $\mathbf{i} \neq \mathbf{j}$, we have*

$$S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap U = \varnothing\,.$$

2. *For all $\mathbf{i},\mathbf{j} \in \Sigma^*$ with $|\mathbf{j}| < |\mathbf{i}|$, we have*

$$\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C) = 0\,.$$

Proof. 1. This follows from Condition 2 of the IOSC.
2. Since $|\mathbf{j}| < |\mathbf{i}|$, there are $\mathbf{j}_0, \mathbf{i}_0 = i_{01} \ldots i_{0n} \in \Sigma^*$ with $n > 0$ and $|\mathbf{j}_0| = |\mathbf{j}|$ such that $\mathbf{i} = \mathbf{j}_0\mathbf{i}_0$. We now divide the proof into two cases.
Case 1: $\mathbf{j} = \mathbf{j}_0$. If $\mathbf{j} = \mathbf{j}_0$, then $S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} = S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{S_{\mathbf{i}_0}U} = S_{\mathbf{j}}^{-1}S_{\mathbf{j}}\overline{S_{\mathbf{i}_0}U} = \overline{S_{\mathbf{i}_0}U} \subseteq \overline{S_{i_{01}}U}$, and we therefore conclude from Condition 5 of the IOSC that $\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C) \leq \nu(\overline{S_{i_{01}}U} \cap C) = 0$.
Case 2: $\mathbf{j} \neq \mathbf{j}_0$. If $\mathbf{j} \neq \mathbf{j}_0$, then Condition 1 and Condition 4 of the IOSC and Part 1 of the lemma imply that $S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C = S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{S_{\mathbf{i}_0}U} \cap C \subseteq S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap \overline{U} = \left(S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap U\right) \cup \left(S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap \partial U\right) = S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap \partial U$. Therefore we conclude from this and Condition 6 of the IOSC that $\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C) \leq \nu(S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap \partial U) = \nu(S_{\mathbf{j}}^{-1}S_{\mathbf{j}_0}\overline{U} \cap \partial U \cap C) \leq \nu(\partial U \cap C) = 0$.
□

We are now ready to prove the main result in this section.

Proposition 4.44. *Assume that the IOSC is satisfied and let U be the open set in the IOSC.*

1. *For all Borel sets $B \subseteq \mathbb{R}^d$, we have $\mu(B) = \mu(B \cap U)$.*
2. *For all $\mathbf{i} \in \Sigma^*$, we have $\mu(\overline{S_{\mathbf{i}}U}) = p_{\mathbf{i}}$.*
3. *For all $\mathbf{i} \in \Sigma^*$, we have $\mu(S_{\mathbf{i}}K_C) = p_{\mathbf{i}}$.*

Proof. 1. We first show that

$$\mu(\overline{U} \setminus U) = 0\,. \tag{4.70}$$

We will now prove (4.70). First we note that if $p = 0$, i.e. if we have (ordinary) self-similar measure, then it is well known that (4.70) holds, c.f. [Gra95]. Thus we now prove (4.70) for $p \neq 0$. For each positive integer n we have

$$\begin{aligned}
\mu(\overline{U} \setminus U) &= \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|=n}} p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U)) + p \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|<n}} p_{\mathbf{i}}\nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U)) \\
&\leq \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|=n}} p_{\mathbf{i}} + p \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|<n}} p_{\mathbf{i}}\nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U)) \\
&= \left(\sum_i p_i\right)^n + p \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|<n}} p_{\mathbf{i}}\nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U)) \\
&= (1-p)^n + p \sum_{\substack{\mathbf{i}\in\Sigma^* \\ |\mathbf{i}|<n}} p_{\mathbf{i}}\nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U) \cap C)\,.
\end{aligned} \tag{4.71}$$

Next, note that for all $\mathbf{i} \in \Sigma^*$ we have

$$S_{\mathbf{i}}^{-1}(\overline{U} \setminus U) \cap (C \cap U) = \varnothing\,. \tag{4.72}$$

We will now prove (4.72). Assume in order to reach a contradiction that (4.72) is not satisfied, i.e. there are $\mathbf{i} \in \Sigma^*$ and $x \in S_{\mathbf{i}}^{-1}(\overline{U}\setminus U)\cap(C\cap U)$. This clearly implies that $S_{\mathbf{i}}x \in (\overline{U}\setminus U)\cap S_{\mathbf{i}}(C\cap U) \subseteq (\overline{U} \setminus U) \cap S_{\mathbf{i}}U \subseteq (\overline{U} \setminus U) \cap U$, yielding the desired contradiction, and completing the proof of (4.72). It follows from (4.72) and Condition 6 of the IOSC that

$$\begin{aligned}
\nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U) \cap C) &= \nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U) \cap (C \cap U)) + \nu(S_{\mathbf{i}}^{-1}(\overline{U} \setminus U) \cap (C \cap \partial U)) \\
&\leq \nu(\varnothing) + \nu(C \cap \partial U) \\
&= 0\,,
\end{aligned} \tag{4.73}$$

for all $\mathbf{i} \in \Sigma^*$.
Combining (4.71) and (4.73) gives

$$\mu(\overline{U} \setminus U) \leq (1-p)^n$$

for all n. Now letting $n \to \infty$ shows that $\mu(\overline{U} \setminus U) = 0$. This completes the proof of (4.70). Finally, since $\operatorname{supp}\mu = K_C \subseteq \overline{U}$ and $\mu(\overline{U} \setminus U) = 0$, we deduce that if B is any Borel set, then $\mu(B) = \mu(B \cap \overline{U}) = \mu(B \cap U)$. This completes the proof of Part 1.

2. We have

$$\begin{aligned}
\mu(\overline{S_{\mathbf{i}}U}) &= \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|=|\mathbf{i}|}} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}) + p \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|<|\mathbf{i}|}} p_{\mathbf{i}}\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}) \\
&= p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}\overline{S_{\mathbf{i}}U}) + \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|=|\mathbf{i}| \\ \mathbf{j}\neq\mathbf{i}}} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}) + p \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|<|\mathbf{i}|}} p_{\mathbf{i}}\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C)\,.
\end{aligned}$$

We conclude from Part 1 that $\mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}) = \mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap U)$ for all $\mathbf{j} \in \Sigma^*$, and so

$$\mu(\overline{S_{\mathbf{i}}U}) = p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}\overline{S_{\mathbf{i}}U}) + \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|=|\mathbf{i}| \\ \mathbf{j}\neq\mathbf{i}}} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap U) + p \sum_{\substack{\mathbf{j}\in\Sigma^* \\ |\mathbf{j}|<|\mathbf{i}|}} p_{\mathbf{i}}\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C)\,. \tag{4.74}$$

Next, it follows from Lemma 4.43 (Part 1) that $S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}\cap U=\varnothing$, and so $\mu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}\cap U)=0$, for all $\mathbf{j}\in\Sigma^*$ with $|\mathbf{j}|=|\mathbf{i}|$ and $\mathbf{j}\neq\mathbf{i}$. It also follows from Lemma 4.43 (Part 2) that $\nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}\cap C)=0$, for all $\mathbf{j}\in\Sigma^*$ with $|\mathbf{j}|<|\mathbf{i}|$. We therefore conclude from (4.74) that

$$\begin{aligned}\mu(\overline{S_{\mathbf{i}}U}) &= p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}\overline{S_{\mathbf{i}}U})\\ &= p_{\mathbf{i}}\mu(\overline{U})\,. \end{aligned}\tag{4.75}$$

Finally, since $\operatorname{supp}\mu=K_C\subseteq\overline{U}$ (by Lemma 4.42) we deduce that $\mu(\overline{U})=1$, and (4.75) therefore implies that

$$\mu(\overline{S_{\mathbf{i}}U})=p_{\mathbf{i}}\,.$$

This completes the proof of Part 2.

3. We have

$$\begin{aligned}p_{\mathbf{i}} &= \mu(\overline{S_{\mathbf{i}}U}) && \text{[by Part 2]}\\ &\geq \mu(S_{\mathbf{i}}K_C) && \text{[by Lemma 4.42]}\\ &= \sum_{\substack{\mathbf{j}\in\Sigma^*\\ |\mathbf{j}|=|\mathbf{i}|}} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}S_{\mathbf{i}}K_C)+p\sum_{\substack{\mathbf{j}\in\Sigma^*\\ |\mathbf{j}|<|\mathbf{i}|}} p_{\mathbf{j}}\nu(S_{\mathbf{j}}^{-1}S_{\mathbf{i}}K_C)\\ &\geq p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}S_{\mathbf{i}}K_C)\\ &= p_{\mathbf{i}}\mu(K_C)\\ &= p_{\mathbf{i}}\,,\end{aligned}$$

whence $\mu(S_{\mathbf{i}}K_C)=p_{\mathbf{i}}$. This completes the proof of Part 3. $\square$

4.3.4 Proof of Theorem 4.28: Preliminary results-part 2

The purpose of this section is to apply the IOSC to prove Proposition 4.47 relating the multifractal spectra of μ to the multifractal spectra of ν . However, we first prove a few auxiliary results.

Lemma 4.45. *Assume that the IOSC satisfied and let U be the open set in the IOSC. For all $\mathbf{i}\in\Sigma^*$ and for all $x\in C\setminus(\cup_i\overline{S_iU}\cup\partial U)$ we have*

$$\begin{aligned}\underline{\dim}_{\mathrm{loc}}(S_{\mathbf{i}}x,\mu) &= \underline{\dim}_{\mathrm{loc}}(x,\nu)\,,\\ \overline{\dim}_{\mathrm{loc}}(S_{\mathbf{i}}x,\mu) &= \overline{\dim}_{\mathrm{loc}}(x,\nu)\,.\end{aligned}$$

Proof. Fix $x\in C\setminus(\cup_i\overline{S_iU}\cup\partial U)$ and $\mathbf{i}\in\Sigma^*$. Since $x\in C\setminus\partial U$ and $C\subseteq U\cup\partial U$, we conclude that $x\in U$. As U is open this implies that there is a positive number $t_{\mathbf{i}}$ such that for all $0<r<t_{\mathbf{i}}$, we have $B(x,r_{\mathbf{i}}^{-1}r)\subseteq U$. It follows from this that

$$B(S_{\mathbf{i}}x,r)=S_{\mathbf{i}}B(x,r_{\mathbf{i}}^{-1}r)\subseteq S_{\mathbf{i}}U$$

for all $0<r<t_{\mathbf{i}}$.
Hence, if $\mathbf{j}\in\Sigma^*$ with $|\mathbf{j}|=|\mathbf{i}|$ and $\mathbf{j}\neq\mathbf{i}$ and $0<r<t_{\mathbf{i}}$ we conclude by Lemma 4.43 (Part 1) that $S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r)\cap U\subseteq S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U}\cap U=\varnothing$, whence

$$\mu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r)\cap U)=0\,.\tag{4.76}$$

Also, if $\mathbf{j} \in \Sigma^*$ with $|\mathbf{j}| < |\mathbf{i}|$, then $S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r) \cap C \subseteq S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C$, whence using Lemma 4.43 (Part 2),

$$\nu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r) \cap C) \le \nu(S_{\mathbf{j}}^{-1}\overline{S_{\mathbf{i}}U} \cap C) = 0\,. \tag{4.77}$$

For $0 < r < t_{\mathbf{i}}$ we now deduce from Proposition 4.44, (4.76) and (4.77) that

$$\begin{aligned}
\mu(B(S_{\mathbf{i}}x,r)) &= \sum_{|\mathbf{j}|=|\mathbf{i}|} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r)) \,+\, p\sum_{|\mathbf{j}|<|\mathbf{i}|} p_{\mathbf{j}}\nu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r))\\
&= p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}B(S_{\mathbf{i}}x,r))\\
&\qquad + \sum_{\substack{\mathbf{j}\in\Sigma^*\\ |\mathbf{j}|=|\mathbf{i}|\\ \mathbf{j}\neq\mathbf{i}}} p_{\mathbf{j}}\mu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r)\cap U) \,+\, p\sum_{\substack{\mathbf{j}\in\Sigma^*\\ |\mathbf{j}|<|\mathbf{i}|}} p_{\mathbf{j}}\nu(S_{\mathbf{j}}^{-1}B(S_{\mathbf{i}}x,r)\cap C)\\
&\qquad\qquad \text{[by Proposition 4.44]}\\
&= p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}B(S_{\mathbf{i}}x,r))\\
&= p_{\mathbf{i}}\mu(S_{\mathbf{i}}^{-1}S_{\mathbf{i}}B(x,r_{\mathbf{i}}^{-1}r))\\
&= p_{\mathbf{i}}\mu(B(x,r_{\mathbf{i}}^{-1}r))\\
&= p_{\mathbf{i}}\mu(B(x,r_{\mathbf{i}}^{-1}r)\cap K_C)\,.
\end{aligned} \tag{4.78}$$

Next, we observe that since $x \in C \setminus \cup_i\overline{S_iU}$ and $\cup_i\overline{S_iU}$ is closed, there is a positive number t_0 such that

$$B(x,r_{\mathbf{i}}^{-1}r) \subseteq \mathbb{R}^d \setminus \bigcup_i \overline{S_iU} \tag{4.79}$$

for all $0 < r < t_0$.
In particular, we deduce from (4.79) and the fact that $K_C \subseteq \overline{U}$, that

$$\begin{aligned}
B(x,r_{\mathbf{i}}^{-1}r)\cap K_C &= B(x,r_{\mathbf{i}}^{-1}r)\cap\left(\bigcup_i S_iK_C \;\cup\; C\right)\\
&\subseteq B(x,r_{\mathbf{i}}^{-1}r)\cap\left(\bigcup_i \overline{S_iU} \;\cup\; C\right)\\
&= B(x,r_{\mathbf{i}}^{-1}r)\cap C\\
&\subseteq B(x,r_{\mathbf{i}}^{-1}r)\cap K_C
\end{aligned}$$

for all $0 < r < t_0$, and so

$$B(x,r_{\mathbf{i}}^{-1}r)\cap K_C = B(x,r_{\mathbf{i}}^{-1}r)\cap C \tag{4.80}$$

for all $0 < r < t_0$. Combining (4.78) and (4.80) now gives

$$\mu(B(S_{\mathbf{i}}x,r)) = p_{\mathbf{i}}\mu(B(x,r_{\mathbf{i}}^{-1}r)\cap C) \tag{4.81}$$

for all $0 < r < \min(t_{\mathbf{i}},t_0)$.
Finally, we observe that if $A \subseteq C \setminus \cup_i\overline{S_iU}$, then

$$\mu(A) = p\nu(A)\,. \tag{4.82}$$

We will now prove (4.82). We first show that if $A \subseteq C \setminus \cup_i\overline{S_iU}$, then $S_i^{-1}A \cap K_C = \varnothing$ for all i. Indeed, otherwise there is an index j and a point $y \in S_j^{-1}A \cap K_C$, whence $S_jy \in A \cap S_jK_C$. But this contradicts the fact that $A \cap S_jK_C \subseteq A \cap \overline{S_jU} \subseteq A \cap (\cup_i\overline{S_iU}) \subseteq (C \setminus \cup_i\overline{S_iU}) \cap (\cup_i\overline{S_iU}) = \varnothing$. The above contradiction shows that $S_i^{-1}A \cap K_C = \varnothing$ for all i. It therefore follows that $\mu(A) = \sum_i p_i\mu(S_i^{-1}A) + p\nu(A) = \sum_i p_i\mu(S_i^{-1}A \cap K_C) + p\nu(A) = p\nu(A)$. This proves (4.82).

It follows from (4.79) that $B(x,r_{\mathbf{i}}^{-1}r)\cap C\subseteq C\setminus\cup_i\overline{S_iU}$, and by applying (4.82) to $A=B(x,r_{\mathbf{i}}^{-1}r)\cap C$, we now conclude that

$$\mu(B(x,r_{\mathbf{i}}^{-1}r)\cap C)=p\nu(B(x,r_{\mathbf{i}}^{-1}r)\cap C) \tag{4.83}$$

for all $0<r<t_0$. Finally, combining (4.81) and (4.83) we conclude that

$$\mu(B(S_{\mathbf{i}}x,r))=pp_{\mathbf{i}}\nu(B(x,r_{\mathbf{i}}^{-1}r)\cap C)=pp_{\mathbf{i}}\nu(B(x,r_{\mathbf{i}}^{-1}r))$$

$0<r<\min(t_0,t_{\mathbf{i}})$. The desired result follows immediately from this. □

Lemma 4.46. *Assume that the IOSC is satisfied and let U be the open set in the IOSC. For all $\alpha\geq 0$ we have*

$$\begin{aligned}\Big\{x\in K_C\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\} &= \Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\\ &\cup\bigcup_{\mathbf{i}\in\Sigma^*} S_{\mathbf{i}}\left\{x\in C\setminus\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(x;\nu)=\alpha\right\}\\ &\cup\bigcup_{\mathbf{i}\in\Sigma^*} S_{\mathbf{i}}\left\{x\in C\cap\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\right\}.\end{aligned}$$

Proof. It follows immediately from Lemma 3.9 that

$$\begin{aligned}\Big\{x\in K_C\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\} &= \Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\\ &\cup\bigcup_{\mathbf{i}\in\Sigma^*}\Big\{x\in S_{\mathbf{i}}C\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}.\end{aligned} \tag{4.84}$$

Also, it is clear that

$$\Big\{y\in S_{\mathbf{i}}C\,\Big|\,\dim_{\mathrm{loc}}(y;\mu)=\alpha\Big\}=S_{\mathbf{i}}\Big\{x\in C\,\Big|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\Big\} \tag{4.85}$$

for all $\mathbf{i}\in\Sigma^*$. Furthermore, it follows from Lemma 4.45 that

$$\begin{aligned}&S_{\mathbf{i}}\Big\{x\in C\,\Big|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\Big\}\\ &=S_{\mathbf{i}}\left\{x\in C\setminus\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\right\}\\ &\qquad\cup S_{\mathbf{i}}\left\{x\in C\cap\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\right\}\\ &=S_{\mathbf{i}}\left\{x\in C\setminus\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(x;\nu)=\alpha\right\}\\ &\qquad\cup S_{\mathbf{i}}\left\{x\in C\cap\left(\bigcup_i\overline{S_iU}\cup\partial U\right)\,\middle|\,\dim_{\mathrm{loc}}(S_{\mathbf{i}}x;\mu)=\alpha\right\}\end{aligned} \tag{4.86}$$

for all $\mathbf{i}\in\Sigma^*$.
The desired result follows by combining (4.84), (4.85) and (4.86). □

We can now state and prove the main result in this section relating the multifractal spectra of μ to the multifractal spectra of ν.

Proposition 4.47. *Assume that the IOSC is satisfied. For all* $\alpha \geq 0$ *we have*

$$\begin{aligned} f_{\mathrm{H},\mu}(\alpha) &= \max\left(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} ,\ f_{\mathrm{H},\nu}(\alpha) \right), \\ f_{\mathrm{P},\mu}(\alpha) &= \max\left(\dim_{\mathrm{P}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} ,\ f_{\mathrm{P},\nu}(\alpha) \right). \end{aligned}$$

Proof. Let U be the open set in the IOSC. It follows immediately from Conditions 5 and 6 of the IOSC that

$$\dim_{\mathrm{H}} \left(C \cap \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \right) = 0 .$$

This and Lemma 4.46 now imply that

$$\begin{aligned} f_{\mathrm{H},\mu}(\alpha) &= \dim_{\mathrm{H}} \left\{ x \in K_C \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \\ &= \max\Bigg(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} , \\ &\qquad \sup_{\mathbf{i} \in \Sigma^*} \dim_{\mathrm{H}} S_{\mathbf{i}} \left\{ x \in C \setminus \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \,\middle|\, \dim_{\mathrm{loc}}(x;\nu) = \alpha \right\} , \\ &\qquad \sup_{\mathbf{i} \in \Sigma^*} \dim_{\mathrm{H}} S_{\mathbf{i}} \left\{ x \in C \cap \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \,\middle|\, \dim_{\mathrm{loc}}(S_{\mathbf{i}} x;\mu) = \alpha \right\} \Bigg) \\ &= \max\Bigg(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} , \\ &\qquad \sup_{\mathbf{i} \in \Sigma^*} \dim_{\mathrm{H}} \left\{ x \in C \setminus \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \,\middle|\, \dim_{\mathrm{loc}}(x;\nu) = \alpha \right\} , \\ &\qquad \sup_{\mathbf{i} \in \Sigma^*} \dim_{\mathrm{H}} \left\{ x \in C \cap \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \,\middle|\, \dim_{\mathrm{loc}}(S_{\mathbf{i}} x;\mu) = \alpha \right\} \Bigg) \\ &= \max\Bigg(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} , \\ &\qquad \dim_{\mathrm{H}} \left\{ x \in C \setminus \left(\bigcup_i \overline{S_i U} \cup \partial U \right) \,\middle|\, \dim_{\mathrm{loc}}(x;\nu) = \alpha \right\} \Bigg) \\ &= \max\Bigg(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} , \\ &\qquad \dim_{\mathrm{H}} \left\{ x \in C \,\middle|\, \dim_{\mathrm{loc}}(x;\nu) = \alpha \right\} \Bigg) \\ &= \max\left(\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} ,\ f_{\mathrm{H},\nu}(\alpha) \right). \end{aligned}$$

This completes the proof. The proof of the formula for $f_{\mathrm{P},\mu}(\alpha)$ is similar. □

Since $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{P}} E$ for any set $E \subseteq \mathbb{R}^d$, Proposition 4.47 shows that in order to prove Theorem 4.28 it suffices to prove the following two inequalities, namely,

$$\beta^*(\alpha) \leq \dim_{\mathrm{H}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\}, \tag{4.87}$$

and

$$\dim_{\mathrm{P}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \leq \beta^*(\alpha). \tag{4.88}$$

The proof of inequality (4.87) will be given in Section 4.3.5 and the proof of inequality (4.88) will be given in Section 4.3.6.

4.3.5 Proof of Theorem 4.28: part 1

The purpose of this setion is to prove the inequality (4.87), i.e.

$$\dim_{\mathrm{H}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \geq \beta^*(\alpha).$$

The proof of this inequality will be given in Proposition 4.51. We now introduce some notation. Let $\Sigma^{\mathbb{N}} = \{1,\ldots,N\}^{\mathbb{N}}$, i.e. $\Sigma^{\mathbb{N}}$ denotes the family of all infinite lists $\mathbf{i} = i_1 i_2 \ldots$ with entries i_j from $\{1,\ldots,N\}$. For $\mathbf{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ and a positive integer n, let $\mathbf{i}|n = i_1 \ldots i_n$. Next, for $\mathbf{i} = i_1 \ldots i_n \in \Sigma^*$, let $[\mathbf{i}]$ denote the cylindre generated by $\mathbf{i}$, i.e. $[\mathbf{i}] = \left\{ \mathbf{j} \in \Sigma^{\mathbb{N}} \,\middle|\, \mathbf{j}|n = \mathbf{i} \right\}$. Also, for a real number q define the probability vector $(Q_i(q))_i$ by

$$Q_i(q) = p_i^q r_i^{\beta(q)},$$

and for $\mathbf{i} = i_1 \ldots i_n \in \Sigma^*$ write $Q_{\mathbf{i}}(q) = Q_{i_1}(q) \ldots Q_{i_n}(q)$ and let $\tilde{\mu}_q$ denote the unique probability measure on $\Sigma^{\mathbb{N}}$ such that

$$\tilde{\mu}_q([\mathbf{i}]) = Q_{\mathbf{i}}(q)$$

for all $\mathbf{i} \in \Sigma^*$. Finally, define $\pi : \Sigma^{\mathbb{N}} \to \mathbb{R}^d$ by

$$\{\, \pi(\mathbf{i}) \,\} = \bigcap_n S_{\mathbf{i}|n} K_\varnothing$$

for $\mathbf{i}$; it is well-known that $\pi(\Sigma^{\mathbb{N}}) = K_\varnothing$.

Proposition 4.48. *Assume that the IOSC is satisfied. For $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$ we have*

$$\liminf_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} \leq \underline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \overline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \limsup_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}}. \tag{4.89}$$

Proof. The second inequality in (4.89) holds trivially. We will now prove the first and the third inequality in (4.89).

Proof of $\liminf_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} \leq \underline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu)$ *for* $\tilde{\mu}_q$*-a.a.* $\mathbf{i} \in \Sigma^{\mathbb{N}}$. Let U be the open set in the IOSC. We first note that since the open set U has the additional property that $U \cap K_\varnothing \neq \varnothing$, then it follows from [Gra95] that

$$\int |\log \mathrm{dist}(\pi(\mathbf{i}), \partial\overline{U})| \, d\tilde{\mu}_q(\mathbf{i}) < \infty. \tag{4.90}$$

For $n \in \mathbb{N} \cup \{0\}$ define $d_n : \Sigma^{\mathbb{N}} \to \mathbb{R}$ by

$$d_n(\mathbf{i}) = \operatorname{dist}\left(\pi(\mathbf{i}) \, , \, \partial \overline{S_{\mathbf{i}|n} U} \right) .$$

Let $S : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ denote the shift map, i.e. $S\,(i_1 i_2 \ldots) = (i_2 i_3 \ldots)$. Note also that for $n \in \mathbb{N}$ we have $S_{\mathbf{i}|n} \pi(S^n \mathbf{i}) = \pi(\mathbf{i})$. Thus

$$\begin{aligned} d_n(\mathbf{i}) &= \operatorname{dist}\left(\pi(\mathbf{i}) \, , \, \partial \overline{S_{\mathbf{i}|n} U} \right) = \operatorname{dist}\left(S_{\mathbf{i}|n} \pi(S^n \mathbf{i}) \, , \, S_{\mathbf{i}|n} \partial \overline{U} \right) \\ &= r_{\mathbf{i}|n} \operatorname{dist}\left(\pi(S^n \mathbf{i}) \, , \, \partial \overline{U} \right) = r_{\mathbf{i}|n} d_0(S^n \mathbf{i}) \, . \end{aligned} \tag{4.91}$$

Since $d_0(\mathbf{i}) = \operatorname{dist}(\pi(\mathbf{i}), \partial \overline{U})$, (4.90) shows that $\log d_0$ is $\tilde{\mu}_q$-integrable, and we therefore conclude that $|\log d_0(\mathbf{i})| < \infty$ for $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$, whence $d_0(\mathbf{i}) > 0$ for $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$. This observation and (4.91) show that

$$d_n(\mathbf{i}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{4.92}$$

for $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$.
Also, since $\log d_0$ is $\tilde{\mu}_q$-integrable, we infer from the ergodic theorem (for the statement and proof of the ergodic theorem see, for example, [Wal82] or any text book on ergodic theory) that

$$\begin{aligned} \frac{1}{n} \log d_0(S^n \mathbf{i}) &= \frac{1}{n} \sum_{k=1}^{n} \log d_0(S^k \mathbf{i}) - \frac{1}{n} \sum_{k=1}^{n-1} \log d_0(S^k \mathbf{i}) \\ &\overset{n \to \infty}{\to} \int \log d_0 \, d\tilde{\mu}_q - \int \log d_0 \, d\tilde{\mu}_q = 0 \end{aligned} \tag{4.93}$$

for $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$.
Finally, it also follows from the ergodic theorem that

$$\frac{1}{n} \log r_{\mathbf{i}|n} \overset{n \to \infty}{\to} a \tag{4.94}$$

for $\tilde{\mu}_q$-a.a. $\mathbf{i} \in \Sigma^{\mathbb{N}}$ where $a = \sum_i Q_i(q) \log r_i$. Namely, we have

$$\frac{1}{n} \log r_{\mathbf{i}|n} = \frac{1}{n} \sum_{k=1}^{n} \log r_{i_k} \, . \tag{4.95}$$

Define $f : \sum^{\mathbb{N}} \to \mathbb{R}$ by

$$f(\mathbf{i}) = \log r_{i_1} \, .$$

Then (4.95) becomes

$$\frac{1}{n} \log r_{\mathbf{i}|n} = \frac{1}{n} \sum_{k=0}^{n-1} f\left(S^k \mathbf{i} \right) \, . \tag{4.96}$$

Letting $n \to \infty$ in (4.96), we obtain

$$\begin{aligned} \frac{1}{n} \log r_{\mathbf{i}|n} &\to \int f d\tilde{\mu}_q \\ &= \sum_i (\log r_i) \, Q_i(q) \, . \end{aligned}$$

Now, fix $\mathbf{i} \in \Sigma^{\mathbb{N}}$ satisfying (4.92), (4.93) and (4.94), and let $r > 0$ be sufficiently small. Since $\overline{U} \supseteq \overline{S_{\mathbf{i}|1}U} \supseteq \overline{S_{\mathbf{i}|2}U} \supseteq \dots$ and $\pi(\mathbf{i}) \in \cap_n \overline{S_{\mathbf{i}|n}U}$, we conclude that $d_1(\mathbf{i}) \geq d_n(\mathbf{i}) \geq \dots$, and we can thus choose a (unique) positive integer $n(\mathbf{i},r)$ such that $d_{n(\mathbf{i},r)}(\mathbf{i}) \leq r \leq d_{n(\mathbf{i},r)-1}(\mathbf{i})$. It follows from the definition of $d_{n(\mathbf{i},r)-1}(\mathbf{i})$ that $B(\pi(\mathbf{i}),r) \subseteq \overline{S_{\mathbf{i}|n(\mathbf{i},r)-1}U}$. This and the fact that $\mu(\overline{S_{\mathbf{i}|n(\mathbf{i},r)-1}U}) = p_{\mathbf{i}|n(\mathbf{i},r)-1}$ (by Proposition 4.44) imply that

$$\mu(B(\pi(\mathbf{i}),r)) \leq \mu(\overline{S_{\mathbf{i}|n(\mathbf{i},r)-1}U}) = p_{\mathbf{i}|n(\mathbf{i},r)-1}\,. \tag{4.97}$$

Also, observe that it follows from the definition of $d_{n(\mathbf{i},r)}(\mathbf{i})$ and (4.91) that

$$r \geq d_{n(\mathbf{i},r)}(\mathbf{i}) = r_{\mathbf{i}|n(\mathbf{i},r)} d_0(S^{n(\mathbf{i},r)}\mathbf{i})\,. \tag{4.98}$$

Finally, combining (4.97) and (4.98) gives

$$\begin{aligned}
\liminf_{r\searrow 0}\frac{\log\mu(B(\pi(\mathbf{i}),r))}{\log r} &\geq \liminf_{r\searrow 0}\frac{\log p_{\mathbf{i}|n(\mathbf{i},r)-1}}{\log r_{\mathbf{i}|n(\mathbf{i},r)} d_0(S^{n(\mathbf{i},r)}\mathbf{i})}\\
&\geq \liminf_{r\searrow 0}\frac{\log r_{\mathbf{i}|n(\mathbf{i},r)-1}}{\log r_{\mathbf{i}|n(\mathbf{i},r)} + \log d_0(S^{n(\mathbf{i},r)}\mathbf{i})}\,\frac{\log p_{\mathbf{i}|n(\mathbf{i},r)-1}}{\log r_{\mathbf{i}|n(\mathbf{i},r)-1}}\\
&= \liminf_{r\searrow 0}\frac{\frac{1}{n(\mathbf{i},r)}\log r_{\mathbf{i}|n(\mathbf{i},r)-1}}{\frac{1}{n(\mathbf{i},r)}\log r_{\mathbf{i}|n(\mathbf{i},r)} + \frac{1}{n(\mathbf{i},r)}\log d_0(S^{n(\mathbf{i},r)}\mathbf{i})}\,\frac{\log p_{\mathbf{i}|n(\mathbf{i},r)-1}}{\log r_{\mathbf{i}|n(\mathbf{i},r)-1}} \qquad (4.99)
\end{aligned} \tag{4.100}$$

However, since $\mathbf{i}$ satisfies (4.92), (4.93) and (4.94), we conclude that

$$\lim_{r\searrow 0}\frac{\frac{1}{n(\mathbf{i},r)}\log r_{\mathbf{i}|n(\mathbf{i},r)-1}}{\frac{1}{n(\mathbf{i},r)}\log r_{\mathbf{i}|n(\mathbf{i},r)} + \frac{1}{n(\mathbf{i},r)}\log d_0(S^{n(\mathbf{i},r)}\mathbf{i})} = \frac{a}{a+0} = 1\,.$$

This and (4.100) show that

$$\liminf_{r\searrow 0}\frac{\log\mu(B(\pi(\mathbf{i}),r))}{\log r} \geq \liminf_{r\searrow 0}\frac{\log p_{\mathbf{i}|n(\mathbf{i},r)-1}}{\log r_{\mathbf{i}|n(\mathbf{i},r)-1}} \geq \liminf_{n}\frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}}$$

for all $\mathbf{i} \in \Sigma^{\mathbb{N}}$ satisfying (4.92), (4.93) and (4.94). The desired result now follows since the set of $\mathbf{i} \in \Sigma^{\mathbb{N}}$ satisfying (4.92), (4.93) and (4.94) has full measure.

Proof of $\overline{\dim}_{\text{loc}}(\pi(\mathbf{i});\mu) \leq \limsup_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}}$ *for all* $\mathbf{i} \in \Sigma^{\mathbb{N}}$. We may clearly assume that $\operatorname{diam} K_C = 1$. For $\mathbf{i} \in \Sigma^{\mathbb{N}}$ and $r > 0$ we may choose a (unique) positive integer $m(\mathbf{i},r)$ such that $r_{\mathbf{i}|m(\mathbf{i},r)} \leq r \leq r_{\mathbf{i}|m(\mathbf{i},r)-1}$. It follows from the definition of $m(\mathbf{i},r)$ that $S_{\mathbf{i}|m(\mathbf{i},r)}K_C \subseteq B(\pi(\mathbf{i}),r)$. This and the fact that $\mu(S_{\mathbf{i}|m(\mathbf{i},r)}K_C) = p_{\mathbf{i}|m(\mathbf{i},r)}$ (by Proposition 4.44) imply that

$$p_{\mathbf{i}|m(\mathbf{i},r)} = \mu(S_{\mathbf{i}|m(\mathbf{i},r)}K_C) \leq \mu(B(\pi(\mathbf{i}),r))\,. \tag{4.101}$$

Also, observe that it follows from the definition of $m(\mathbf{i},r)-1$ that

$$r \leq r_{\mathbf{i}|m(\mathbf{i},r)-1} \leq \frac{1}{r_{\min}} r_{\mathbf{i}|m(\mathbf{i},r)}\,. \tag{4.102}$$

Combining (4.101) and (4.102) gives

$$\limsup_{r\searrow 0}\frac{\log\mu(B(\pi(\mathbf{i}),r))}{\log r} \leq \limsup_{r\searrow 0}\frac{\log p_{\mathbf{i}|m(\mathbf{i},r)}}{\log\frac{1}{r_{\min}}r_{\mathbf{i}|m(\mathbf{i},r)}} = \limsup_{r\searrow 0}\frac{\log p_{\mathbf{i}|m(\mathbf{i},r)}}{\log r_{\mathbf{i}|m(\mathbf{i},r)}} \leq \limsup_{m}\frac{\log p_{\mathbf{i}|m}}{\log r_{\mathbf{i}|m}}\,.$$

This completes the proof. $\square$

Define the probability vector $\mathbf{p}_0 = (p_{0,i})_i$ by

$$p_{0,i} = \frac{p_i}{\sum_j p_j} .$$

For $\mathbf{i} = i_1 \dots i_n \in \Sigma^*$ write $p_{0,\mathbf{i}} = p_{0,i_1} \dots p_{0,i_n}$.

Lemma 4.49. *For all* $\mathbf{i} \in \Sigma^{\mathbb{N}}$ *and all* n *we have* $\frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = -\frac{\log \sum_i p_i}{\frac{\log r_{\mathbf{i}|n}}{n}} + \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}}$.

Proof. First note that

$$p_{0,\mathbf{i}|n} = \frac{p_{\mathbf{i}|n}}{\left(\sum_j p_j\right)^n} .$$

Using this, we have

$$\begin{aligned}
\frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} &= \frac{\log p_{\mathbf{i}|n} - n \log \sum_j p_j}{\log r_{\mathbf{i}|n}} \\
&= \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} - \frac{\log \sum_j p_j}{\frac{\log r_{\mathbf{i}|n}}{n}} .
\end{aligned}$$

□

Lemma 4.50. *We have,*

1. *If* $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$, *then* $\beta^*(\alpha) > 0$.
2. *If* $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, *then* $\beta^*(\alpha) = -\infty$.

Proof. First recall that

$$1 = \sum_i p_i^q r_i^{\beta(q)} . \tag{4.103}$$

From this we immediately have that

$$\begin{aligned}
\lim_{q \to \infty} \beta(q) &= -\infty \\
\lim_{q \to -\infty} \beta(q) &= \infty .
\end{aligned}$$

Let

$$\alpha_i = \frac{\log p_i}{\log r_i} , \quad \alpha_{\min} = \min_i \frac{\log p_i}{\log r_i} \quad \text{and} \quad \alpha_{\max} = \max_i \frac{\log p_i}{\log r_i} \tag{4.104}$$

Also, let

$$f(\alpha) = \beta^*(\alpha) = \inf_{-\infty < q < \infty} \{\beta(q) + \alpha q\} \tag{4.105}$$

Since β is strictly convex (provided that α_i are not the same for all i), for a given α the infinium in (4.105) is attained at a unique $q = q(\alpha)$. This occurs when $\alpha = \alpha(q) = -\beta'(q)$. Thus

$$f(\alpha) : (\alpha(\infty), \alpha(-\infty)) \to \mathbb{R}^+ .$$

We will now show that

$$\alpha(\infty) = \alpha_{\min} \quad \text{and} \quad \alpha(-\infty) = \alpha_{\max} .$$

We have,

$$
\begin{aligned}
1 &= \sum_{\alpha_i=\alpha_{\min}} p_i^q r_i^{\beta(q)} + \sum_{\alpha_i>\alpha_{\min}} p_i^q r_i^{\beta(q)} \\
&= \sum_{\alpha_i=\alpha_{\min}} \left(p_i r_i^{-\alpha_{\min}}\right)^q r_i^{\beta(q)+\alpha_{\min}q} + \sum_{\alpha_i>\alpha_{\min}} \left(p_i r_i^{-\alpha_{\min}}\right)^q r_i^{\beta(q)+\alpha_{\min}q} \\
&= \sum_{\alpha_i=\alpha_{\min}} r_i^{\beta(q)+\alpha_{\min}q} + \sum_{\alpha_i>\alpha_{\min}} \left(p_i r_i^{-\alpha_{\min}}\right)^q r_i^{\beta(q)+\alpha_{\min}q}\,. \qquad (4.106)
\end{aligned}
$$

Observe that $\beta(q)+\alpha_{\min}q$ is nonincreasing, since

$$
\begin{aligned}
\beta'(q)+\alpha_{\min} &= -\frac{\sum_i p_i^q r_i^{\beta(q)}\log p_i}{\sum_i p_i^q r_i^{\beta(q)}\log r_i} + \alpha_{\min} \\
&= -\frac{\sum_i \alpha_i p_i^q r_i^{\beta(q)}\log r_i + \alpha_{\min}\sum_i p_i^q r_i^{\beta(q)}\log r_i}{\sum_i p_i^q r_i^{\beta(q)}\log r_i} \\
&= \frac{\sum_i (\alpha_{\min}-\alpha_i)\, p_i^q r_i^{\beta(q)}\log r_i}{\sum_i p_i^q r_i^{\beta(q)}\log r_i} \le 0
\end{aligned}
$$

Thus, we have

$$\lim_{q\to\infty}\left(\beta(q)+\alpha_{\min}q\right) = -\infty \qquad (4.107)$$

or, we have

$$\lim_{q\to\infty}\left(\beta(q)+\alpha_{\min}q\right) = e \quad \text{for some } e\in\mathbb{R} \qquad (4.108)$$

If (4.107) holds, then taking limit in (4.106) as $q\to\infty$, we obtain $1=\infty$.
Therefore (4.108) must hold. Then taking the limit in (4.106) as $q\to\infty$, we obtain

$$1 = \sum_{\alpha_i=\alpha_{\min}} r_i^e\,. \qquad (4.109)$$

To find the asymptotic behavior of α, we first observe that.

$$
\begin{aligned}
\alpha(q) &= \frac{\sum_i p_i^q r_i^{\beta(q)}\log p_i}{\sum_i p_i^q r_i^{\beta(q)}\log r_i} \\
&= \frac{\sum_{\alpha_i=\alpha_{\min}} r_i^{\beta(q)+\alpha_{\min}q}\log p_i + \sum_{\alpha_i>\alpha_{\min}} r_i^{\beta(q)+\alpha_i q}\log p_i}{\sum_{\alpha_i=\alpha_{\min}} r_i^{\beta(q)+\alpha_{\min}q}\log r_i + \sum_{\alpha_i>\alpha_{\min}} r_i^{\beta(q)+\alpha_i q}\log r_i} \\
&= \frac{\sum_{\alpha_i=\alpha_{\min}} r_i^{\beta(q)+\alpha_{\min}q}\log p_i + \sum_{\substack{\alpha_i=\alpha_{\min}+\epsilon_i\\ \epsilon_i>0}} r_i^{\beta(q)+\alpha_{\min}q} r_i^{\epsilon_i q}\log p_i}{\sum_{\alpha_i=\alpha_{\min}} r_i^{\beta(q)+\alpha_{\min}q}\log r_i + \sum_{\substack{\alpha_i=\alpha_{\min}+\epsilon_i\\ \epsilon_i>0}} r_i^{\beta(q)+\alpha_{\min}q} r_i^{\epsilon_i q}\log r_i}\,. \qquad (4.110)
\end{aligned}
$$

Using (4.109) and letting $q\to\infty$ in (4.110), we obtain

$$\alpha(\infty) = \alpha_{\min}\,.$$

Similarly, one can show that

$$\alpha(-\infty) = \alpha_{\max}\,.$$

$\square$

We can now state and prove the main result in this section.

Proposition 4.51. *Assume that the IOSC is satisfied. For all* $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$ *we have*

$$\dim_{\mathrm{H}} \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \geq \beta^*(\alpha)\,.$$

Proof. As in the proof of Lemma 4.50, let $\alpha = -\beta'$. An argument from the proof of Lemma 4.50 shows that $\{\alpha(q) \,|\, q \in \mathbb{R}\} = (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$, and we therefore find $q \in \mathbb{R}$ such that $\alpha = \alpha(q)$. It follows easily by implicit differentiation that

$$\alpha(q) = -\beta'(q) = \frac{\sum_i Q_i(q)\log p_i}{\sum_i Q_i(q)\log r_i}\,.$$

Write

$$\begin{aligned}
\Pi &= \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \liminf_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} \leq \underline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \overline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \limsup_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} \right\},\\
\Gamma &= \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log r_{\mathbf{i}|n}}{n} = \sum_i Q_i(q)\log r_i \right\},\\
\Xi &= \left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = \frac{\sum_i Q_i(q)\log p_{0,i}}{\sum_i Q_i(q)\log r_i} \right\}.
\end{aligned}$$

Lemma 4.49 implies that

$$\begin{aligned}
\left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} &= \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha(q) \right\}\\
&\supseteq \left\{ x \in K_{\varnothing} \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha(q) \right\} \cap \pi(\Pi) \cap \pi(\Gamma)\\
&\supseteq \pi\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = \alpha(q) \right\} \cap \pi(\Pi) \cap \pi(\Gamma)\\
&= \pi\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = -\frac{\log \sum_i p_i}{\sum_i Q_i(q)\log r_i} + \alpha(q) \right\}\\
&\qquad \cap \pi(\Pi) \cap \pi(\Gamma)\\
&= \pi\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = -\frac{\log \sum_i p_i}{\sum_i Q_i(q)\log r_i} + \frac{\sum_i Q_i(q)\log p_i}{\sum_i Q_i(q)\log r_i} \right\}\\
&\qquad \cap \pi(\Pi) \cap \pi(\Gamma)\\
&= \pi\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = \frac{\sum_i Q_i(q)\log p_{0,i}}{\sum_i Q_i(q)\log r_i} \right\}\\
&\qquad \cap \pi(\Pi) \cap \pi(\Gamma)\\
&= \pi(\Xi) \cap \pi(\Pi) \cap \pi(\Gamma)\,.
\end{aligned} \tag{4.111}$$

Let $\mu_q = \tilde{\mu}_q \circ \pi^{-1}$. It follows immediately from the ergodic theorem that

$$\mu_q(\pi(\Gamma)) \geq \tilde{\mu}_q(\Gamma) = \tilde{\mu}_q\left\{ \mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log r_{\mathbf{i}|n}}{n} = \sum_i Q_i(q)\log r_i \right\} = 1\,,$$

$$\mu_q(\pi(\Xi)) \quad \geq \quad \tilde{\mu}_q(\Xi) = \tilde{\mu}_q\left\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \lim_n \frac{\log p_{0,\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} = \frac{\sum_i Q_i(q) \log p_{0,i}}{\sum_i Q_i(q) \log r_i}\right\} = 1\,, \qquad (4.112)$$

and it follows from Proposition 4.48 that

$$\begin{aligned} \mu_q(\pi(\Pi)) \quad &= \quad \tilde{\mu}_q(\pi^{-1}\pi(\Pi)) \\ &\geq \quad \tilde{\mu}_q\left\{\mathbf{i} \in \Sigma^{\mathbb{N}} \,\middle|\, \liminf_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}} \leq \underline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \overline{\dim}_{\mathrm{loc}}(\pi(\mathbf{i});\mu) \leq \limsup_n \frac{\log p_{\mathbf{i}|n}}{\log r_{\mathbf{i}|n}}\right\} = 1\,. \end{aligned} \qquad (4.113)$$

Combining (4.111), (4.112) and (4.113) shows that

$$\mu_q\left\{x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha\right\} = 1\,.$$

We conclude from this that

$$\dim_{\mathrm{H}}\left\{x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha\right\} \geq \inf_{\mu_q(E)=1} \dim_{\mathrm{H}} E\,. \qquad (4.114)$$

Finally, it is well-known that

$$\inf_{\mu_q(E)=1} \dim_{\mathrm{H}} E = \frac{\sum_i Q_i(q) \log Q_i(q)}{\sum_i Q_i(q) \log r_i}\,; \qquad (4.115)$$

see, for example, [AP96] or [Fal97]. Combining (4.114) and (4.115) shows that

$$\begin{aligned} \dim_{\mathrm{H}}\left\{x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha\right\} \quad &\geq \quad \frac{\sum_i Q_i(q) \log Q_i(q)}{\sum_i Q_i(q) \log r_i} \\ &= \quad q\frac{\sum_i Q_i(q) \log p_i}{\sum_i Q_i(q) \log r_i} + \beta(q) \\ &= \quad q\alpha(q) + \beta(q) \\ &= \quad q\alpha + \beta(q) \\ &\geq \quad \beta^*(\alpha)\,. \end{aligned}$$

This completes the proof. □

4.3.6 Proof of Theorem 4.28: part 2

The purpose of this section is to prove the inequality (4.88), i.e.

$$\dim_{\mathrm{P}}\left\{x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha\right\} \leq \beta^*(\alpha)\,.$$

The proof of this inequality will be given in Proposition 4.53. The next lemma is a slight modification of a result due to Hutchinson [Hut81] (cf. also [Fal90]) and the proof is therefore omitted.

Lemma 4.52. *Let $r, c_1, c_2 > 0$, and let $(V_i)_i$ be a family of open disjoint subsets of $\mathbb{R}^d$ such that V_i contains a ball of radius $c_1 r$ and is contained in a ball of radius $c_2 r$. Then*

$$\Big|\{i \mid B(x,r) \cap \overline{V_i} \neq \varnothing\}\Big| \leq \left(\frac{1+2c_2}{c_1}\right)^d$$

for all $x \in \mathbb{R}^d$.

Here by $\Big|\{i \mid B(x,r) \cap \overline{V_i} \neq \varnothing\}\Big|$ we mean the cardinality of the set $\{i \mid B(x,r) \cap \overline{V_i} \neq \varnothing\}$. We use this notation for the rest of this section.

We can now state and prove the main result in this section.

Proposition 4.53. *Assume that the IOSC is satisfied. Fix $\alpha \geq 0$.*

1. *Let $(r_n)_n$ be a sequence in $(0,1)$ for which there exists a constant $c \in (0,1)$ such that*

$$\begin{gathered} r_n \searrow 0\,, \\ cr_n < r_{n+1} < r_n \quad \textit{for all } n, \\ \textstyle\sum_n r_n^\varepsilon < \infty \quad \textit{for all } \varepsilon > 0. \end{gathered}$$

(For example, me may take $r_n = a^n$ for $a \in (0,1)$.) Let $q \in \mathbb{R}$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Write

$$\begin{aligned} \Delta_{n,\varepsilon} &= \bigcap_{k \geq n} \left\{ x \in K_\varnothing \,\middle|\, \alpha - \varepsilon \leq \frac{\log \mu(B(x,r_k))}{\log r_k} \leq \alpha + \varepsilon \right\}, \\ \Pi^q_{n,\varepsilon} &= \inf_{r_n > 0} \left\{ \sum_i (2\delta_i)^{q\alpha + \beta(q) + (1+|q|)\varepsilon} \,\middle|\, \big(B(x_i,\delta_i)\big)_i \textit{ is a centered } r_n\textit{-packing of } \Delta_{n,\varepsilon} \right\}. \end{aligned}$$

Then

$$\Pi^q_{n,\varepsilon} < \infty\,.$$

2. *If $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$, then*

$$\dim_{\mathrm{P}} \left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} \leq \beta^*(\alpha)\,.$$

3. *If $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, then*

$$\left\{ x \in K_\varnothing \,\middle|\, \dim_{\mathrm{loc}}(x;\mu) = \alpha \right\} = \varnothing\,.$$

Proof. 1. We may clearly assume that $\operatorname{diam} K_C = 1$. Let $(B(x_i,\delta_i))_i$ be a centered r_n-packing of $\Delta_{n,\varepsilon}$. For each i let n_i denote the unique positive integer such that

$$r_{n_i} \leq \delta_i < r_{n_i - 1}\,.$$

Observe that $r_{n_i} \leq \delta_i \leq r_n$, whence

$$n_i \geq n\,. \tag{4.116}$$

Next we prove the following five claims.

Claim 1. *There is a constant* $c_1 > 0$ *such that* $(2\delta_i)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} \leq c_1 r_{n_i}^{q\alpha+\beta(q)+(1+|q|)\varepsilon}$ *for all* i.

Proof of Claim 1. This follows immediately from the definitions. This completes the proof of Claim 1.

Claim 2. *We have* $r_{n_i}^{q\alpha+|q|\varepsilon} \leq \mu(B(x_i, r_{n_i}))^q$ *for all* i.

Proof of Claim 2. Since $(B(x_i,\delta_i))_i$ is a centered r_n-packing of $\Delta_{n,\varepsilon}$, we deduce that $x_i \in \Delta_{n,\varepsilon}$. This implies that $\alpha - \varepsilon \leq \frac{\log \mu(B(x_i,r_k))}{\log r_k} \leq \alpha + \varepsilon$ for all $k \geq n$. In particular, we conclude that $r_k^{\alpha+\varepsilon} \leq \mu(B(x_i, r_k)) \leq r_k^{\alpha-\varepsilon}$ for all $k \geq n$. However, since $n_i \geq n$ (by (4.116)), the desired conclusion follows from this inequality. This completes the proof Claim 2.

In order to state Claim 3 and Claim 4 we make the following definitions. For $\mathbf{i} \in \Sigma^*$ and positive integers i and m write

$$\begin{aligned}
\Gamma_i &= \Big\{ \mathbf{i} \in \Sigma^* \,\Big|\, r_{\mathbf{i}} \leq r_{n_i} < r_{\mathbf{i}|\,|\mathbf{i}|-1}\,,\ S_{\mathbf{i}}K_C \cap B(x_i, r_{n_i}) \neq \varnothing \Big\}\,,\\
I_{\mathbf{i},m} &= \Big\{ i \in \mathbb{N} \,\Big|\, n_i = m\,,\ S_{\mathbf{i}}K_C \cap B(x_i, r_m) \neq \varnothing \Big\}\,.
\end{aligned} \tag{4.117}$$

Claim 3. *There is a constant* $c_2 > 0$ *such that* $|\Gamma_i| \leq c_2$ *for all* i.

Proof of Claim 3. Let U be the open set in the IOSC. Now put

$$\Pi_i = \Big\{ \mathbf{i} \in \Sigma^* \,\Big|\, r_{\mathbf{i}} \leq r_{n_i} < r_{\mathbf{i}|\,|\mathbf{i}|-1}\,,\ S_{\mathbf{i}}\overline{U} \cap B(x_i, r_{n_i}) \neq \varnothing \Big\}$$

for $i \in \mathbb{N}$. It follows from Lemma 4.42 that $S_{\mathbf{i}}K_C \subseteq S_{\mathbf{i}}\overline{U}$ for all $\mathbf{i} \in \Sigma^*$, whence

$$\Gamma_i \subseteq \Pi_i\,.$$

Since U is non-empty, bounded and open, there are two numbers $s_1, s_2 > 0$ such that U contains a ball of radius s_1 and is contained in a ball of radius s_2. Hence, if $\mathbf{i} \in \Pi_i$, then $S_{\mathbf{i}}U$ contains a ball of radius $r_{\mathbf{i}}s_1$, and since $r_{\mathbf{i}}s_1 \geq r_{\mathbf{i}|\,|\mathbf{i}|-1} r_{\min} s_1 \geq (r_{\min}s_1) r_{n_i}$, we deduce that $S_{\mathbf{i}}U$ contains a ball of radius $(r_{\min}s_1)r_{n_i}$. Similarly, if $\mathbf{i} \in \Pi_i$, then $S_{\mathbf{i}}U$ is contained in a ball of radius $r_{\mathbf{i}}s_2$, and since $r_{\mathbf{i}}s_2 \leq s_2 r_{n_i}$, we deduce that $S_{\mathbf{i}}U$ is contained in a ball of radius $s_2 r_{n_i}$. Since also the sets $(S_{\mathbf{i}}U)_{\mathbf{i}\in\Pi_i}$ are pairwise disjoint (because $S_iU \cap S_jU = \varnothing$ for $i \neq j$), it therefore follows from Lemma 4.52 that

$$|\Gamma_i| \leq |\Pi_i| \leq \left(\frac{1+2s_2}{r_{\min}s_1} \right)^d .$$

This completes the proof of Claim 3.

Claim 4. *There is a constant* $c_3 > 0$ *such that* $|I_{\mathbf{i},m}| \leq c_3$ *for all* $\mathbf{i} \in \Sigma^*$ *and all* $m \in \mathbb{N}$ *with* $r_{\mathbf{i}} \leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}$.

Proof of Claim 4. Choose any positive real number $s > 0$ such that $K_C \subseteq B(0,s)$. Now put

$$J_{\mathbf{i},m} = \Big\{ i \in \mathbb{N} \,\Big|\, n_i = m\,,\ B(S_{\mathbf{i}}0, r_m s) \cap \overline{B(x_i, r_m)} \neq \varnothing \Big\}$$

for $\mathbf{i} \in \Sigma^*$ and $m \in \mathbb{N}$. Next, fix $\mathbf{i} \in \Sigma^*$ and $m \in \mathbb{N}$ with $r_{\mathbf{i}} \leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}$. Since $K_C \subseteq B(0,s)$, it follows that $S_{\mathbf{i}}K_C \subseteq B(S_{\mathbf{i}}0, r_{\mathbf{i}}s)$, whence

$$\begin{aligned} I_{\mathbf{i},m} &\subseteq \left\{ i \in \mathbb{N} \,\middle|\, n_i = m \;,\; B(S_{\mathbf{i}}0, r_{\mathbf{i}}s) \cap \overline{B(x_i, r_m)} \neq \varnothing \right\} \\ &\subseteq \left\{ i \in \mathbb{N} \,\middle|\, n_i = m \;,\; B(S_{\mathbf{i}}0, r_m s) \cap \overline{B(x_i, r_m)} \neq \varnothing \right\} \\ &= J_{\mathbf{i},m} \,. \end{aligned}$$

It is also clear that $B(x_i, r_m)$ contains a ball of radius $r_m = \frac{1}{s} s r_m$, and that $B(x_i, r_m)$ is contained in a ball of radius $r_m = \frac{1}{s} s r_m$. Since also the sets $(B(x_i, r_m))_{i \in J_{\mathbf{i},m}}$ are pairwise disjoint (because $(\, B(x_i, \delta_i)\,)_i$ is a packing and $B(x_i, r_m) = B(x_i, r_{n_i}) \subseteq B(x_i, \delta_i)$ for $i \in J_{\mathbf{i},m}$ since $r_{n_i} \leq \delta_i$), it therefore follows from Lemma 4.52 that

$$|I_{\mathbf{i},m}| \leq |J_{\mathbf{i},m}| \leq \left(\frac{1 + 2\frac{1}{s}}{\frac{1}{s}} \right)^d = (2+s)^d \,.$$

This completes the proof of Claim 4.

Claim 5. *There is a constant* $c_4 > 0$ *such that* $r_m^{\beta(q)} \leq c_4 r_{\mathbf{i}}^{\beta(q)}$ *for all* $\mathbf{i} \in \Sigma^*$ *and all* $m \in \mathbb{N}$ *with* $r_{\mathbf{i}} \leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}$.
Proof of Claim 5. This follows immediately from the definitions. This completes the proof of Claim 5.

We conclude from Claim 1 and Claim 2 that

$$\begin{aligned} \sum_i (2\delta_i)^{q\alpha + \beta(q) + (1+|q|)\varepsilon} &\leq c_1 \sum_i r_{n_i}^{q\alpha+\beta(q)+(1+|q|)\varepsilon} \\ &\leq c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} \, \mu(B(x_i, r_{n_i}))^q \,. \end{aligned} \tag{4.118}$$

Next, since $x_i \in \Delta_{n,\varepsilon} \subseteq K_\varnothing$ we can choose $\mathbf{i}_i \in \Sigma^{\mathbb{N}}$ such that $x_i = \pi(\mathbf{i}_i)$. Let m_i denote the unique positive integer such that $r_{\mathbf{i}_i|m_i} \leq r_{n_i} < r_{\mathbf{i}_i|\,m_i-1}$. Observe that since $x_i = \pi(\mathbf{i}_i) \in S_{\mathbf{i}_i|m_i}K_\varnothing \subseteq S_{\mathbf{i}_i|m_i}K_C$ and $\operatorname{diam} S_{\mathbf{i}_i|m_i}K_C = r_{\mathbf{i}_i|m_i} \operatorname{diam} K_C = r_{\mathbf{i}_i|m_i} \leq r_{n_i}$ (recall that we are assuming $\operatorname{diam} K_C = 1$), we deduce that $S_{\mathbf{i}_i|m_i}K_C \subseteq B(x_i, r_{n_i})$. In particular, it follows from this that if $q < 0$, then

$$\mu(B(x_i, r_{n_i}))^q \leq \mu(S_{\mathbf{i}_i|m_i}K_C)^q \,. \tag{4.119}$$

Also note that

$$\mathbf{i}_i|m_i \in \Gamma_i \,. \tag{4.120}$$

We will now prove (4.120). Indeed, it is clear that $r_{\mathbf{i}_i|m_i} \leq r_{n_i} < r_{\mathbf{i}_i|\,m_i-1}$, and since $x_i = \pi(\mathbf{i}_i) \in S_{\mathbf{i}_i|m_i}K_\varnothing \subseteq S_{\mathbf{i}_i|m_i}K_C$ and $x_i \in B(x_i, r_{n_i})$, we conclude that $S_{\mathbf{i}_i|m_i}K_C \cap B(x_i, r_{n_i}) \neq \varnothing$. This proves (4.120).
Finally, note that $B(x_i, r_{n_i}) \subseteq \cup_{\mathbf{i} \in \Gamma_i} S_{\mathbf{i}}K_C$. In particular, it follows from this that if $0 \leq q$, then

$$\mu(B(x_i, r_{n_i}))^q \leq \mu\left(\bigcup_{\mathbf{i} \in \Gamma_i} S_{\mathbf{i}}K_C \right)^q . \tag{4.121}$$

Combining (4.119), (4.120) and (4.121) and using the fact that $\mu(S_{\mathbf{i}}K_C) = p_{\mathbf{i}}$ for all $\mathbf{i} \in \Sigma^*$ (by Proposition 4.44) we now deduce from (4.118) that

$$
\begin{aligned}
\sum_i (2\delta_i)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} &\leq \begin{cases} c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon}\, \mu(S_{\mathbf{i}_i|m_i} K_C)^q & \text{for } q<0; \\ c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon}\, \mu\left(\bigcup_{\mathbf{i}\in\Gamma_i} S_{\mathbf{i}} K_C\right)^q & \text{for } 0\leq q; \end{cases} \\
&\leq \begin{cases} c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon}\, \mu(S_{\mathbf{i}_i|m_i} K_C)^q & \text{for } q<0; \\ c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} \left(\sum_{\mathbf{i}\in\Gamma_i} \mu(S_{\mathbf{i}} K_C)\right)^q & \text{for } 0\leq q; \end{cases} \\
&= \begin{cases} c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon}\, p_{\mathbf{i}_i|m_i}^q & \text{for } q<0; \\ c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} \left(\sum_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}\right)^q & \text{for } 0\leq q; \end{cases} \\
&\leq \begin{cases} c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} \sup_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q & \text{for } q<0; \\ c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} |\Gamma_i|^q \sup_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q & \text{for } 0\leq q; \end{cases} \\
&\leq c_1 \sum_i r_{n_i}^{\beta(q)+\varepsilon} |\Gamma_i|^{|q|} \sup_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q\,. \qquad (4.122)
\end{aligned}
$$

Next using Claim 3 and the fact that $n_i \geq n$ (by (4.116)) we dedue from (4.122) that

$$
\begin{aligned}
\sum_i (2\delta_i)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} &\leq c_1\, c_2^{|q|} \sum_i r_{n_i}^{\beta(q)+\varepsilon} \sup_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q \\
&= c_1\, c_2^{|q|} \sum_{m\geq n} \sum_{\substack{i \\ n_i=m}} r_{n_i}^{\beta(q)+\varepsilon} \sup_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q \\
&\leq c_1\, c_2^{|q|} \sum_{m\geq n} \sum_{\substack{i \\ n_i=m}} r_m^{\beta(q)+\varepsilon} \sum_{\mathbf{i}\in\Gamma_i} p_{\mathbf{i}}^q \\
&= c_1\, c_2^{|q|} \sum_{m\geq n} r_m^{\beta(q)+\varepsilon} \sum_{\substack{i \\ n_i=m}} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1} \\ S_{\mathbf{i}}K_C\cap B(x_i,r_m)\neq\varnothing}} p_{\mathbf{i}}^q \\
&= c_1\, c_2^{|q|} \sum_{m\geq n} r_m^{\beta(q)+\varepsilon} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}}} \sum_{\substack{i \\ n_i=m \\ S_{\mathbf{i}}K_C\cap B(x_i,r_m)\neq\varnothing}} p_{\mathbf{i}}^q \\
&= c_1\, c_2^{|q|} \sum_{m\geq n} r_m^{\beta(q)+\varepsilon} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}}} p_{\mathbf{i}}^q\, |I_{\mathbf{i},m}|\,. \qquad (4.123)
\end{aligned}
$$

Using Claim 4 and Claim 5 we conclude from (4.123) that

$$
\begin{aligned}
\sum_i (2\delta_i)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} &\leq c_1\, c_2^{|q|}\, c_3 \sum_{m\geq n} r_m^{\beta(q)+\varepsilon} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}}} p_{\mathbf{i}}^q \\
&\leq c_1\, c_2^{|q|}\, c_3\, c_4 \sum_{m\geq n} r_m^{\varepsilon} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}}} r_{\mathbf{i}}^{\beta(q)} p_{\mathbf{i}}^q \\
&= c_1\, c_2^{|q|}\, c_3\, c_4 \sum_{m\geq n} r_m^{\varepsilon} \sum_{\substack{\mathbf{i}\in\Sigma^* \\ r_{\mathbf{i}}\leq r_m < r_{\mathbf{i}|\,|\mathbf{i}|-1}}} \tilde{\mu}_q([\mathbf{i}])\,; \qquad (4.124)
\end{aligned}
$$

recall that the measure $\tilde{\mu}_q$ is defined at the beginning of Section 4.3.5.

Finally, using the fact that the sets $\big([\mathbf{i}]\big)_{\mathbf{i}\in\Sigma^*\,,\,r_{\mathbf{i}}\le r_m<r_{\mathbf{i}|\,|\mathbf{i}|-1}}$ are pairwise disjoint, we conclude from (4.124) that

$$\begin{aligned}
\sum_i (2\delta_i)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} &\le c_1\,c_2^{|q|}\,c_3\,c_4 \sum_{m\ge n} r_m^{\varepsilon}\,\tilde\mu_q\left(\bigcup_{\substack{\mathbf{i}\in\Sigma^*\\ r_{\mathbf{i}}\le r_m<r_{\mathbf{i}|\,|\mathbf{i}|-1}}}[\mathbf{i}]\right)\\
&\le c_1\,c_2^{|q|}\,c_3\,c_4 \sum_{m\ge n} r_m^{\varepsilon}\\
&\le c_1\,c_2^{|q|}\,c_3\,c_4 \sum_{m} r_m^{\varepsilon}\\
&= c_0\,,
\end{aligned}$$

where $c_0=c_1c_2^{|q|}c_3c_4\sum_m r_m^{\varepsilon}$. This completes the proof of Part 1 of Proposition 4.53.

We now turn towards the proof of Part 2 and Part 3 of Proposition 4.53. However, we first introduce some notation. For brevity write

$$\Delta=\Big\{x\in K_\varnothing\,\Big|\,\dim_{\mathrm{loc}}(x;\mu)=\alpha\Big\}\,.$$

Next, let $(r_n)_n$ be a sequence in $(0,1)$ and let $c\in(0,1)$ be a real number satisfying the conditions in Part 1 of Proposition 4.53. For a positive integer n and $\varepsilon>0$, let $\Delta_{n,\varepsilon}$ be defined as in Part 1 of Proposition 4.53, i.e.

$$\Delta_{n,\varepsilon}=\bigcap_{k\ge n}\left\{x\in K_\varnothing\,\middle|\,\alpha-\varepsilon\le\frac{\log\mu(B(x,r_k))}{\log r_k}\le\alpha+\varepsilon\right\}.$$

Next observe that since $cr_n<r_{n+1}<r_n$ for all n, we conclude that

$$\begin{aligned}
\Delta &= \left\{x\in K_\varnothing\,\middle|\,\lim_{r\searrow 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha\right\}\\
&= \left\{x\in K_\varnothing\,\middle|\,\lim_{n\to\infty}\frac{\log\mu(B(x,r_n))}{\log r_n}=\alpha\right\}\\
&\subseteq \bigcup_n\bigcap_{k\ge n}\left\{x\in K_\varnothing\,\middle|\,\alpha-\varepsilon\le\frac{\log\mu(B(x,r_k))}{\log r_k}\le\alpha+\varepsilon\right\}\\
&= \bigcup_n \Delta_{n,\varepsilon}\,,
\end{aligned}\tag{4.125}$$

for all $\varepsilon>0$.

Proof of Part 2 of Proposition 4.53. Let $q\in\mathbb{R}$. We must now prove that

$$\dim_{\mathrm{P}}\Delta\le q\alpha+\beta(q)\,.\tag{4.126}$$

In order to prove (4.126), inclusion (4.125) shows that it suffices to prove that

$$\dim_{\mathrm{P}}\Delta_{n,\varepsilon}\le q\alpha+\beta(q)+(1+|q|)\varepsilon\tag{4.127}$$

for all positive integers n and all $\varepsilon > 0$. Therefore, fix a positive integer n and $\varepsilon > 0$. We will now prove (4.127). Let $\Pi^q_{n,\varepsilon}$ be defined as in Part 1. Also, recall that for $t, r > 0$, we let $\overline{\mathcal{P}}^t$ denote the t-dimensional packing pre-measure and we let $\overline{\mathcal{P}}^t_r$ denote the r-approximative t-dimensional packing pre-measure. Since $\alpha \in (\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i})$, we conclude from Lemma 4.50 that $q\alpha + \beta(q) + (1+|q|)\varepsilon \geq \beta^*(\alpha) + (1+|q|)\varepsilon > 0$, and so $\overline{\mathcal{P}}^{q\alpha+\beta(q)+(1+|q|)\varepsilon}_{r_n}(\Delta_{n,\varepsilon}) = \Pi^q_{n,\varepsilon}$. It now follows from Part 1 that

$$\begin{aligned} \mathcal{P}^{q\alpha+\beta(q)+(1+|q|)\varepsilon}(\Delta_{n,\varepsilon}) &\leq \overline{\mathcal{P}}^{q\alpha+\beta(q)+(1+|q|)\varepsilon}(\Delta_{n,\varepsilon}) \\ &\leq \overline{\mathcal{P}}^{q\alpha+\beta(q)+(1+|q|)\varepsilon}_{r_n}(\Delta_{n,\varepsilon}) \\ &= \Pi^q_{n,\varepsilon} \\ &< \infty, \end{aligned}$$

whence $\dim_{\mathrm{P}} \Delta_{n,\varepsilon} \leq q\alpha + \beta(q) + (1+|q|)\varepsilon$.

Proof of Part 3 of Proposition 4.53. Since $\alpha \notin [\min_i \frac{\log p_i}{\log r_i}, \max_i \frac{\log p_i}{\log r_i}]$, we conclude from Lemma 4.50 that $\beta^*(\alpha) < 0$ and we can thus find $q \in \mathbb{R}$ such that $q\alpha + \beta(q) < 0$. It follows from this that we can choose $\varepsilon > 0$ such that $q\alpha + \beta(q) + (1+|q|)\varepsilon < 0$.
We will now prove that $\Delta = \varnothing$. Assume in order to reach a contradiction that $\Delta \neq \varnothing$. We can therefore find $x \in \Delta$. In particular, it follows from (4.125) that there is a positive integer n such that $x \in \Delta_{n,\varepsilon}$. Hence, for each $\delta > 0$ with $\delta \leq r_n$, the ball $B(x,\delta)$ is a centred r_n-packing of $\Delta_{n,\varepsilon}$, and we therefore conclude from Part 1 that

$$(2\delta)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} \leq \Pi^q_{n,\varepsilon}.$$

Since $q\alpha + \beta(q) + (1+|q|)\varepsilon < 0$ and $0 < \delta \leq r_n$ was arbitrary, this implies that

$$\infty = \sup_{0<\delta\leq r_n} (2\delta)^{q\alpha+\beta(q)+(1+|q|)\varepsilon} \leq \Pi^q_{n,\varepsilon} < \infty. \tag{4.128}$$

The desired contradiction follows immediately from (4.128). This completes the proof of Proposition 4.53. □

4.4 Open problems for multifractal analysis of inhomogeneous self-similar measures

4.4.1 Open problems for L^q spectra and Rényi dimensions of inhomogeneous self-similar measures

It is quite unsatisfactory that our results for L^q spectra and Rényi dimensions are obtained under the assumption that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint. It is natural to ask if the results are true assuming only the appropriate version of the standard Open Set Condition. Namely, assuming Inhomogeneous Open Set Condition (IOSC) stated in Section 4.3.1.

Question 4.54. *Are the results in Section 4.2.1 true if the IOSC is satisfied?*

Question 4.55. *Assume that the sets $(S_1K_C, \ldots, S_NK_C, C)$ are pairwise disjoint (or simply assume that the IOSC is satisfied).*

- *Is it true that*
$$\overline{\tau}_\mu(q) = \max\Big(\beta(q), \overline{\tau}_\nu(q)\Big),$$
for all $q \in \mathbb{R}$?

- *Can we find an upper bound for* $\underline{\tau}_\mu(q)$ *for all* $q \in \mathbb{R}$ *?*

Question 4.56. *Are the results in Section 4.2.2 true if the IOSC is satisfied?*

5 Fourier transforms of inhomogeneous self-similar measures

5.1 Preliminaries: Fourier transforms of measures

One of the reasons to study Fourier transforms of a measure is that the behaviour of Fourier transform of a measure gives the information about the continuity of a measure. Namely, the faster the Fourier transform of a measure tends to zero the more regular the measure is. Thus, for example, since the Fourier transforms of discrete measures do not tend to zero (see Theorem 5.1 below), these measures are not regular. In analysing the Fourier transform of a measure we are interested in investigating not only the asymptotic behaviour of the Fourier transform itself but its asymptotic behaviour in an average sense, i.e. we want to analyse the following integral $\int\limits_{B(0,R)} |\widehat{\mu}(x)|^2\,dx$, or more generally $\int\limits_{B(0,R)} |\widehat{\mu}(x)|^q\,dx$, for $q \in (0,\infty)$. This leads to the following general definitions of Fourier dimensions. Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\widehat{\mu}$ denote the Fourier transform of μ . Recall that $\widehat{\mu}\,(x) = \int e^{i\langle y|x\rangle} d\mu\,(y)$ for $x \in \mathbb{R}^d$. Then for $q \in (0,\infty]$, we define the q 'th upper Fourier dimension $\overline{\Delta}_q(\mu)$ and we define the q 'th lower Fourier dimension $\underline{\Delta}_q(\mu)$ of μ as follows. For $q < \infty$, we put

$$\overline{\Delta}_q(\mu) = \limsup_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} |\widehat{\mu}(x)|^q\,dx\right)^{\frac{1}{q}}}{-\log R}, \tag{5.1}$$

$$\underline{\Delta}_q(\mu) = \liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} |\widehat{\mu}(x)|^q\,dx\right)^{\frac{1}{q}}}{-\log R}, \tag{5.2}$$

where $\mathcal{L}^d$ denotes d -dimensional Lebesgue measure, and for $q=\infty$, we put

$$\overline{\Delta}_\infty(\mu) = \limsup_{R\to\infty} \frac{\log \sup_{|x|\geq R} |\widehat{\mu}(x)|}{-\log R}, \tag{5.3}$$

$$\underline{\Delta}_\infty(\mu) = \liminf_{R\to\infty} \frac{\log \sup_{|x|\geq R} |\widehat{\mu}(x)|}{-\log R}. \tag{5.4}$$

We now state a well known Wiener's Theorem [Wie33] describing the asymptotic behaviour of $\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} |\widehat{\mu}(x)|^2\,dx$ and therefore providing the information about $\overline{\Delta}_2(\mu)$ and $\underline{\Delta}_2(\mu)$.

Theorem 5.1. [**[Wie33]**, **see also** **[Str94]**]. *Let δ_x denote the Dirac measure supported at a point x . Suppose $\mu = \mu_1 + \mu_2$ where $\mu_1 = \sum_j c_j \delta_{a_j}$ is discrete and μ_2 is continuous. Then we have*

$$\lim_{R\to\infty} \frac{1}{\mathcal{L}^d(B(0,R))} \int\limits_{B(0,R)} |\widehat{\mu}(x)|^2\,dx = \sum_j |c_j|^2 .$$

Hence, it is clear that if μ is a probability measure and has a discrete part then $\overline{\Delta}_2(\mu) = \underline{\Delta}_2(\mu) = 0$.

Remark. We note further that if a probability measure μ has a discrete part then for $q \geq 1$ we have

$$\overline{\Delta}_q(\mu) = \underline{\Delta}_q(\mu) = \overline{\Delta}_\infty(\mu) = \underline{\Delta}_\infty(\mu) = 0 .$$

Thus, we are interested in computing Fourier dimensions (for $q \geq 1$) of probabilty measures which are continuous.

5.2 Fourier transforms of inhomogeneous self-similar measures

During the past 10 years there has been an enormous interest in investigating Fourier dimensions of (ordinary) self-similar measures satisfying (2.2) and there is a huge body of literature discussing this problem, see, for example, Bluhm [Blu99], Hu [Hu01], Hu & Lau [HL02], Lau [Lau92, Lau95], Lau & Wang [LW93], Strichartz [Str90a, Str90b, Str93a, Str93b]. Almost all of these papers concentrate their study on the 2'nd Fourier dimension of self-similar measures [Lau92, Lau95, LW93, Str90a, Str90b, Str93a, Str93b] or on the infinity Fourier dimension of self-similar measures [Blu99, Hut81, HL02]. Continuing this line of investigation, in this section we will study the 2'nd Fourier dimension and the infinity Fourier dimension of inhomogeneous self-similar measures. In Theorem 5.2 we obtain bounds for the infinity Fourier dimension of an inhomogeneous self-similar measure and in Theorem 5.4 we obtain bounds for the 2'nd Fourier dimension of an inhomogeneous self-similar measure. Finally, in Section 5.2.3 we present a number of applications of our results. In particular, non-linear self-similar measures introduced and investigated by Glickenstein & Strichartz [GS96] are special cases of inhomogeneous self-similar measures, and as an application of our main results we obtain simple proofs of generalizations of Glickenstein & Strichartz's results on the asymptotic behaviour of the Fourier transforms of non-linear self-similar measures.

5.2.1 Main results

From now we fix an inhomogeneous self-similar measure μ satisfying

$$\mu = \sum_j p_j \mu \circ S_j^{-1} + p\nu \,, \tag{5.5}$$

where $(p_1, \ldots, p_N, p)$ is a probability vector and $S_j x = r_j R_j x + a_j$ for $0 < r_j < 1$, R_j is an orthogonal matrix and $a_j \in \mathbb{R}^d$. Before stating our main results it is useful to introduce the following terminology.

Definition. Equicontractive. We will say that the equicontractive condition is satisfied if all the contraction ratios $r_1, \ldots, r_N$ coincide, i.e. if $r_1 = \ldots = r_N$.

We will now state the first of our main results providing a lower bound for the infinity Fourier dimension of an inhomogeneous self-similar measure.

Theorem 5.2. [The $\underline{\Delta}_\infty(\mu)$ dimension of an inhomogeneous self-similar measure μ]. *Define s and t by*

$$\begin{aligned} \sum_j p_j r_{\min}^{-s} &= 1 \quad i.e. \quad s = \frac{\log(1-p)}{\log r_{\min}}\,, \\ \sum_j p_j r_j^{-t} &= 1\,, \end{aligned}$$

where $r_{\min} = \min_j r_j$. Then $0 \leq s \leq t$, and

$$\underline{\Delta}_\infty(\mu) \geq \begin{cases} \underline{\Delta}_\infty(\nu) & \text{if } 0 \leq \underline{\Delta}_\infty(\nu) < s; \\ \frac{s\underline{\Delta}_\infty(\nu)}{s+\underline{\Delta}_\infty(\nu)} & \text{if } s \leq \underline{\Delta}_\infty(\nu) \leq t; \\ t & \text{if } t < \underline{\Delta}_\infty(\nu). \end{cases} \tag{5.6}$$

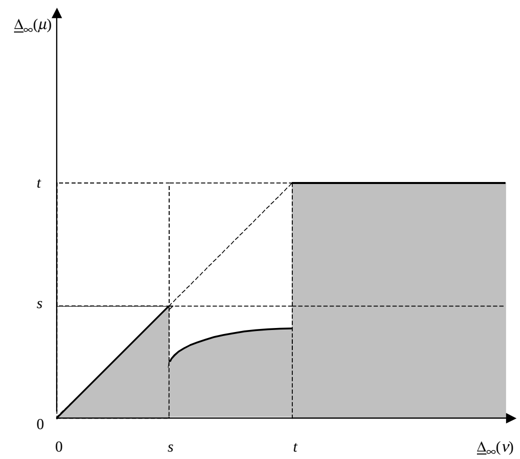

Figure 5.2.1:
The bold lines separating the shaded and the unshaded regions represents the graph of the function $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = x$ for $0 \le x < s$, $f(x) = \frac{sx}{s+x}$ for $s \le x \le t$, and $f(x) = t$ for $t < x$. For each value of $\underline{\Delta}_\infty(\nu)$, the number $f(\underline{\Delta}_\infty(\nu))$ is a lower bound for $\underline{\Delta}_\infty(\mu)$, i.e. $\underline{\Delta}_\infty(\mu)$ lies in the unshaded region above $f(\underline{\Delta}_\infty(\nu))$.

Theorem 5.2 is proved in Section 5.2.4 The reader is referred to Figure 5.2.1 for a graphical illustration of the inequalities in (5.6). In the equicontractive case the lower bound for $\underline{\Delta}_\infty(\mu)$ simplifies considerably. This is the content of the next corollary.

Corollary 5.3. *Assume that $r_1 = \ldots = r_N$. Then $s = t$, and*

$$\underline{\Delta}_\infty(\mu) \ge \min(t, \underline{\Delta}_\infty(\nu)). \tag{5.7}$$

In order to obtain a lower bound for the 2'nd Fourier dimension we must assume that the Open Set Condition (OSC) is satisfied. Recall that the OSC says that there exists an open, non-empty and bounded subset U of $\mathbb{R}^d$ with $\cup_j S_j(U) \subseteq U$ and $S_j(U) \cap S_k(U) = \varnothing$ for all $k \ne j$.

Theorem 5.4. **[The $\underline{\Delta}_2(\mu)$ dimension of an inhomogeneous self-similar measure μ].** *Assume that the OSC is satisfied. Assume further that $r_1 = \ldots = r_N$ and $R_1 = \ldots = R_N$. Note that if the equicontractive condition is satisfied then $s = t$ and*

$$\sum_j p_j r^{-t} = \sum_j p_j r^{-s} = 1 \quad i.e. \quad s = t = \frac{\log(1-p)}{\log r}.$$

Then

$$\underline{\Delta}_2(\mu) \ge \min(t, \underline{\Delta}_2(\nu)).$$

Theorem 5.4 is proved in Section 5.2.5

5.2.2 Open problems and conjectures for Fourier dimensions of inhomogeneous self-similar measures

We have decided to put open problems and conjectures after stating our main results for the following reasons. Namely, below (see Section 5.2.3) we will present a number of examples which will support our conjectures and will give affirmative answers to some of our open questions.

Corollary 5.3 leads us to believe that the same result might be true in the nonequicontractive case. This suggests the following conjecture.

Conjecture 5.5. *For all choices of* $r_1, \ldots, r_N$ *we have,*

$$\underline{\Delta}_\infty(\mu) \geq \min(\, t \,,\, \underline{\Delta}_\infty(\nu)\,)\,. \tag{5.8}$$

The 2'nd Fourier dimension $\underline{\Delta}_2(\mu_0)$ of a self-similar measure μ_0 satisfying (2.2) has been studied in [Str90b] and investigated further in [Lau95, LW93, Str93a, Str93b]. In particular, the following results are proved in [Str90b]. Let $S_j : \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, N$ be contracting similarities and write r_j for the contracting ratio of S_j. Let $(p_1, \ldots, p_N)$ be a probability vector and let μ_0 be the self-similar measure satisfying (2.2). Finally, let t_0 and u_0 be defined by

$$\sum_j p_j r_j^{-t_0} = 1\,, \quad \sum_j p_j^2 r_j^{-u_0} = 1\,,\,,$$

and note that $\frac{u_0}{2} \geq t_0$. In view of Theorem 5.4 one would expect the following lower bound for $\underline{\Delta}_2(\mu_0)$ to hold, namely, $\underline{\Delta}_2(\mu_0) \geq t_0$. In fact, in [Str90b] it is proved that that if the OSC is satisfied, then the following better lower bound for $\underline{\Delta}_2(\mu_0)$ holds, namely,

$$\underline{\Delta}_2(\mu_0) \geq \frac{u_0}{2}\,.$$

It is also proved in [Str90b] that if the equicontracting condition and some further conditions are satisfied, then

$$\underline{\Delta}_2(\mu_0) = \frac{u_0}{2}\,. \tag{5.9}$$

In view of the above remarks, it is natural to conjecture that the result in Theorem 5.4 can be improved as follows.

Conjecture 5.6. *Assume that the OSC is satisfied. Assume further that* $r_1 = \ldots = r_N = r$ *and* $R_1 = \ldots = R_N$ *. Recall that if the equicontractive condition is satisfied, then* $s = t$ *and*

$$\sum_j p_j r^{-t} = \sum_j p_j r^{-s} = 1 \quad i.e. \quad s = t = \frac{\log(1-p)}{\log r}\,.$$

Define u *by*

$$\sum_j p_j^2 r^{-u} = 1\,.$$

It is easily seen that $\frac{u}{2} \geq t$ *, and we now conjecture that Theorem 5.4 can be improved as follows,*

$$\underline{\Delta}_2(\mu) \geq \min\left(\,\frac{u}{2}\,,\, \underline{\Delta}_2(\nu)\,\right)\,. \tag{5.10}$$

It is also natural to ask for lower bounds of $\underline{\Delta}_2(\mu)$ in the nonequicontractive case. Indeed, it is not difficult to see what Conjecture 5.6 looks like in the nonequicontractive case.

Conjecture 5.7. *Assume that the OSC is satisfied. For all choices of $r_1, \ldots, r_N$ we define u by*

$$\sum_j p_j^2 r_j^{-u} = 1 .$$

We conjecture that

$$\underline{\Delta}_2(\mu) \geq \min\left(\frac{u}{2}, \underline{\Delta}_2(\nu)\right) .$$

It follows from Corollary 5.3 and Theorem 5.4 that in the equicontractive case the lower bounds for the infinity Fourier dimension $\underline{\Delta}_\infty(\mu)$ and the 2'nd Fourier dimension $\underline{\Delta}_2(\mu)$ satisfy analogous equations. It is natural to ask if the same result may hold for an arbitrary q'th Fourier dimension of μ.

Question 5.8. *Assume that the OSC is satisfied. Assume further that $r_1 = \ldots = r_N$ and $R_1 = \ldots = R_N$. Is it true that*

$$\underline{\Delta}_q(\mu) \geq \min\left(t, \underline{\Delta}_q(\nu)\right) \tag{5.11}$$

for all $q > 0$? Is (5.11) true even if the equicontractive condition is not satisfied?

If q is an even integer, an extension of the arguments given below in the proof of Theorem 5.4 may lead to this result. However, it appears to us that new ideas are needed if q is not an integer. One of the possibilities to obtain lower bounds for $\underline{\Delta}_2(\mu)$ in the nonequicontractive case is to use techniques from Renewal Theory. Indeed, methods from Renewal Theory were introduced into the study of fractal measures by Lalley [Lal88, Lal91] and have subsequently been used in [Lau95, LW93, Str93a] to investigate the 2'nd Fourier dimension $\underline{\Delta}_2(\mu_0)$ for (ordinary) self-similar measures μ_0 satisfying (2.2) in the nonequicontractive case. In fact, [Lau95, LW93, Str93a] also obtain very precise information about the rate of convergence of $\frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)}(|\widehat{\mu}_0(x)|^2\,dx)^{\frac{1}{2}}\right.}{-\log R}$ to $\underline{\Delta}_2(\mu_0)$ as $R \to \infty$. This suggests that Renewal Theory can also be used in analyzing Fourier dimensions of inhomogeneous self-similar measures.

All results mentioned so far have provided lower bounds for the Fourier dimensions $\underline{\Delta}_q(\mu)$ for $q = 2, \infty$. It is clearly of interest to obtain upper bounds, or even exact values, for those dimensions. We also wonder if the lower bounds in Theorem 5.2 and Conjecture 5.6 are, in fact, exact values. This and (5.9) suggest the following problem and question.

Problem 5.9. *Assume that the OSC is satisfied. Find an upper bound for $\underline{\Delta}_q(\mu)$ for $q = 2, \infty$ (or for all q).*

Question 5.10. *Assume that the OSC is satisfied. Assume further that $r_1 = \ldots = r_N$ and $R_1 = \ldots = R_N$. Is it true that*

$$\underline{\Delta}_\infty(\mu) = \min\left(t, \underline{\Delta}_\infty(\nu)\right)$$

and

$$\underline{\Delta}_2(\mu) = \min\left(\frac{u}{2}, \underline{\Delta}_2(\nu)\right) ?$$

5.2.3 Examples

Next we consider some examples of inhomogeneous self-similar measures and their lower Fourier dimensions, including examples which show that a number of the conjectures in Section 5.2.2 are satisfied in various specific cases.

Example 5.11. **[Non-linear self-similar measures. Part 1.]**

We consider probability measures μ satisfying the following nonlinear self-similar identity

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + \sum_{j=1}^{M} q_j (\mu * \mu) \circ T_j^{-1} , \tag{5.12}$$

where $(p_1, \ldots, p_N, q_1, \ldots, q_M)$ is a probability vector, $S_j x = r_j R_j x + a_j$ for $0 < r_j < 1$, R_j is an orthogonal matrix and $a_i \in \mathbb{R}^d$, and $T_j x = \rho_j P_j x + b_j$ for $0 < \rho_j < \frac{1}{2}$, P_j is an orthogonal matrix and $b_j \in \mathbb{R}^d$; the existence and uniqueness of measures μ satisfying (5.12) is proved in [GS96]. Without loss of generality we may clearly assume that μ is not supported on any $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. Indeed, if this is not the case, then the construction of μ takes place on a $(d-1)$-dimensional affine subspace Γ, say, of $\mathbb{R}^d$ and the extra dimension (orthogonal to Γ) is superfluous. Hence, by successively removing dimensions, we may assume that d is chosen such that μ is not supported on any $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$.
Glickenstein & Strichartz [GS96] also analyzed the asymptotic behaviour of the Fourier transform of μ. We note that measures μ satisfying the nonlinear self-similar identity in (5.12) can be viewed as inhomogeneous self-similar measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N, p, \nu)$ where

$$p = 1 - \sum_{j=1}^{N} p_j \quad \text{and} \quad \nu = \sum_{j=1}^{M} \frac{q_j}{1 - \sum_{k=1}^{N} p_k} (\mu * \mu) \circ T_j^{-1} .$$

We will now apply Theorem 5.2 and Theorem 5.4 to obtain simple proofs of generalizations of the results from [GS96] giving lower bounds for the infinity Fourier dimension and the 2'nd Fourier dimension of μ.

A. The infinity Fourier dimension of μ. We first discuss the infinity Fourier dimension of μ. Before analyzing this example further it is useful to make the following two observations.

Observation 5.12. *We have*

$$\underline{\Delta}_{\infty}(\nu) \geq 2\underline{\Delta}_{\infty}(\mu) . \tag{5.13}$$

Proof. First we note that for $x \in \mathbb{R}^d$

$$\begin{aligned}
\widehat{\nu}\,(x) &= \left(\sum_{j=1}^{M} \frac{q_j}{p} (\mu * \mu) \circ T_j^{-1} \right)^{\widehat{}}(x) \\
&= \sum_{j=1}^{M} \frac{q_j}{p} ((\mu * \mu) \circ T_j^{-1})^{\widehat{}}(x) \\
&= \sum_{j=1}^{M} \frac{q_j}{p} e^{i\langle b_j | x \rangle} (\mu * \mu)^{\widehat{}}(T_j^* x) \\
&= \sum_{j=1}^{M} \frac{q_j}{p} e^{i\langle b_j | x \rangle} \widehat{\mu}\,(T_j^* x)^2 ,
\end{aligned} \tag{5.14}$$

where $T_j^* = \rho_j P_j^*$ and P_j^* is the conjugate transpose of P_j. It follows from (5.14) that

$$\begin{aligned}\underline{\Delta}_\infty(\nu) &= \liminf_{R\to\infty} \frac{\log\left(\sup_{|x|\geq R} \left| \sum_{j=1}^M \frac{q_j}{p} e^{i\langle b_j | x\rangle} \widehat{\mu}(T_j^* x)^2 \right| \right)}{-\log R} \\ &\geq \liminf_{R\to\infty} \frac{\log\left(\sum_{j=1}^M \frac{q_j}{p} \sup_{|x|\geq R} |\widehat{\mu}(T_j^* x)|^2 \right)}{-\log R}. \end{aligned} \tag{5.15}$$

Fix $\varepsilon > 0$. It now follows from the definition of $\underline{\Delta}_\infty(\mu)$ that there exists a constant $c > 0$ such that $\sup_{|x|\geq R} |\widehat{\mu}(x)| \leq c|R|^{-(\underline{\Delta}_\infty(\mu)-\varepsilon)}$ for all $R > 0$, whence $\sup_{|x|\geq R} |\widehat{\mu}(T_j^* x)|^2 \leq c^2(\rho_j R)^{-2(\underline{\Delta}_\infty(\mu)-\varepsilon)}$ for all j. Combining this and (5.15), we conclude that

$$\begin{aligned}\underline{\Delta}_\infty(\nu) &\geq \liminf_{R\to\infty} \frac{\log\left(\sum_{j=1}^M \frac{q_j}{p} c^2 (\rho_j R)^{-2(\underline{\Delta}_\infty(\mu)-\varepsilon)} \right)}{-\log R} \\ &= \liminf_{R\to\infty} \frac{\log\left(C R^{-2(\underline{\Delta}_\infty(\mu)-\varepsilon)} \right)}{-\log R} \\ &= 2(\underline{\Delta}_\infty(\mu) - \varepsilon),\end{aligned}$$

where $C = \sum_{j=1}^M \frac{q_j}{p} c^2 (\rho_j)^{-2(\underline{\Delta}_\infty(\mu)-\varepsilon)}$. Letting $\varepsilon \to 0$ we obtain (5.13). □

Observation 5.13. *We have*

$$\underline{\Delta}_\infty(\mu) > 0.$$

Proof. This is simply a restatement of [[GS96], Lemma 3.1] and the proof is therefore omitted. □

Since d is chosen such that μ is not supported on any $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$ it follows from [GS96] that

$$\underline{\Delta}_\infty(\mu) \geq t. \tag{5.16}$$

We will now show that Glickenstein & Strichartz's result (5.16) implies that Conjecture 5.5 is true. Indeed, it follows from (5.13) and (5.16) that $\underline{\Delta}_\infty(\nu) \geq 2\underline{\Delta}_\infty(\mu) \geq 2t$, whence $\min(t, \underline{\Delta}_\infty(\nu)) = t$. Using (5.16) once more, we conclude from this that

$$\underline{\Delta}_\infty(\mu) \geq t = \min(t, \underline{\Delta}_\infty(\nu)).$$

This shows that (5.16) implies that Conjecture 5.5 is true in this case.
Next, we show that in the equicontractive case, Corollary 5.3 implies Glickenstein & Strichartz's result (5.16). In the equicontractive case we can use Corollary 5.3 and inequality (5.13) to obtain $\underline{\Delta}_\infty(\mu) \geq \min(t, \underline{\Delta}_\infty(\nu)) \geq \min(t, 2\underline{\Delta}_\infty(\mu))$. Since also $\underline{\Delta}_\infty(\mu) > 0$ (by Observation 5.13), we see from this inequality that $\min(t, 2\underline{\Delta}_\infty(\mu)) = t$, whence

$$\underline{\Delta}_\infty(\mu) \geq \min(t, 2\underline{\Delta}_\infty(\mu)) = t.$$

Thus Corollary 5.3 implies Glickenstein & Strichartz's result (5.16) in the equicontractive case.

We now consider the general nonequicontractive case. Since $\underline{\Delta}_\infty(\nu) \geq 2\underline{\Delta}_\infty(\mu)$ (by (5.13)), Theorem 5.2 implies that in the nonequicontractive case we have

$$\underline{\Delta}_\infty(\mu) \geq \begin{cases} 2\underline{\Delta}_\infty(\mu) & \text{if } 0 \leq \underline{\Delta}_\infty(\nu) < s; \\ \frac{2s\underline{\Delta}_\infty(\mu)}{s+2\underline{\Delta}_\infty(\mu)} & \text{if } s \leq \underline{\Delta}_\infty(\nu) \leq t; \\ t & \text{if } t < \underline{\Delta}_\infty(\nu), \end{cases}$$

which simplifies to

$$\underline{\Delta}_\infty(\mu) \quad = \quad 0 \qquad \text{if } 0 \leq \underline{\Delta}_\infty(\nu) < s; \tag{5.17}$$

$$\underline{\Delta}_\infty(\mu) \quad \geq \quad \begin{cases} \frac{s}{2} & \text{if } s \leq \underline{\Delta}_\infty(\nu) \leq t; \\ t & \text{if } t < \underline{\Delta}_\infty(\nu). \end{cases} \tag{5.18}$$

However, since $\underline{\Delta}_\infty(\mu) > 0$ (by Observation 5.13), we deduce from (5.17) that $s \leq \underline{\Delta}_\infty(\nu)$. Also, if $s \leq \underline{\Delta}_\infty(\nu) \leq t$, then it follows from (5.13) that $2\underline{\Delta}_\infty(\mu) \leq \underline{\Delta}_\infty(\nu) \leq t$, whence $\underline{\Delta}_\infty(\mu) \leq \frac{t}{2}$. In view of those remarks, (5.17) and (5.18) simplify to: we have $s \leq \underline{\Delta}_\infty(\nu)$ and

$$\begin{aligned} \tfrac{t}{2} \geq \underline{\Delta}_\infty(\mu) \geq \tfrac{s}{2} \qquad & \text{if } s \leq \underline{\Delta}_\infty(\nu) \leq t, \\ \underline{\Delta}_\infty(\mu) \geq t \qquad & \text{if } t < \underline{\Delta}_\infty(\nu). \end{aligned}$$

Of course, it follows from Glickenstein & Strichartz's result (5.16) that $\underline{\Delta}_\infty(\mu) \geq t$, and it therefore follows from the above that we must have $t < \underline{\Delta}_\infty(\nu)$; however, we cannot deduce this from Theorem 5.2. This again suggests that Theorem 5.2 can be improved for $s \leq \underline{\Delta}_\infty(\nu) \leq t$ as outlined in Conjecture 5.5.

B. The 2'nd Fourier dimension of μ. Next we analyze the 2'nd Fourier dimension of μ. In order to apply Theorem 5.4 to analyze $\underline{\Delta}_2(\mu)$, we will make two further assumptions, namely that $r_1 = \ldots = r_N$ and that $R_1 = \ldots = R_N$. However, before analyzing the 2'nd Fourier dimension it is useful to make the following observation.

Observation 5.14. *For all $q > 1$, we have*

$$\underline{\Delta}_q(\nu) \quad \geq \quad 2\underline{\Delta}_{2q}(\mu)\,, \tag{5.19}$$

$$2\underline{\Delta}_{2q}(\mu) \quad > \quad \underline{\Delta}_q(\mu)\,. \tag{5.20}$$

Proof. We first prove (5.19). It follows from (5.14) and Minkowski's inequality that

$$\begin{aligned} \underline{\Delta}_q(\nu) \quad &= \quad \liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} \left|\sum_{j=1}^M \frac{q_j}{p} e^{i\langle b_j|x\rangle}\widehat{\mu}(T_j^* x)^2\right|^q dx\right)^{\frac{1}{q}}}{-\log R} \\ &\geq \quad \liminf_{R\to\infty} \frac{\log \sum_{j=1}^M \frac{q_j}{p}\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} \left|\widehat{\mu}(T_j^* x)\right|^{2q} dx\right)^{\frac{1}{q}}}{-\log R}\,. \end{aligned} \tag{5.21}$$

Fix $\varepsilon > 0$. Then there exists a constant $c > 0$ such that $(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(x)|^{2q}\,dx)^{\frac{1}{2q}} \leq cR^{-(\underline{\Delta}_{2q}(\mu)-\varepsilon)}$ for all $R > 0$, whence

$$\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} \left|\widehat{\mu}(T_j^* x)\right|^{2q}\,dx\right)^{\frac{1}{q}} \leq c^2(\rho_j R)^{-2(\underline{\Delta}_{2q}(\mu)-\varepsilon)}$$

Combining this and (5.21), we conclude that

$$\begin{aligned}
\underline{\Delta}_q(\nu) &\geq \liminf_{R\to\infty} \frac{\log \sum_{j=1}^M \frac{q_j}{p} c^2 (\rho_j R)^{-2(\underline{\Delta}_{2q}(\mu)-\varepsilon)}}{-\log R} \\
&= \liminf_{R\to\infty} \frac{\log C R^{-2(\underline{\Delta}_{2q}(\mu)-\varepsilon)}}{-\log R} \\
&= 2(\underline{\Delta}_{2q}(\mu)-\varepsilon)\,,
\end{aligned}$$

where $C = \sum_{j=1}^M \frac{q_j}{p} c^2 \rho_j^{-2(\underline{\Delta}_{2q}(\mu)-\varepsilon)}$. Letting $\varepsilon \to 0$ we obtain (5.19).
Next we prove (5.20). In [[GS96], Lemma 3.1] it was shown that $|\widehat{\mu}(x)| < 1$ for all $x \neq 0$. This and the continuity of $\widehat{\mu}$ implies that

$$\begin{aligned}
2\underline{\Delta}_{2q}(\mu) &= 2\liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} |\widehat{\mu}(x)|^{2q}\,dx\right)^{\frac{1}{2q}}}{-\log R} \\
&> 2\liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int\limits_{B(0,R)} |\widehat{\mu}(x)|^{q}\,dx\right)^{\frac{1}{2q}}}{-\log R} \\
&= \underline{\Delta}_q(\mu)\,.
\end{aligned}$$

This completes the proof of (5.20). $\square$

We can now use the above Observation 5.14 and Theorem 5.4 to obtain

$$\underline{\Delta}_2(\mu) \geq \min(\,t\,,\,\underline{\Delta}_2(\nu)\,) \geq \min(\,t\,,\,2\underline{\Delta}_4(\mu)\,) > \min(\,t\,,\,\underline{\Delta}_2(\mu)\,)\;.$$

Thus

$$\underline{\Delta}_2(\mu) \geq t\,.$$

This completes Example 5.11.

Example 5.15. [Non-linear self-similar measures. Part 2.]

Now we consider measures satisfying a more general nonlinear self-similar identity. Namely, we consider probability measures μ satisfying the following nonlinear self-similar identity

$$\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1} + \sum_{j=1}^M q_j (\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}}) \circ T_j^{-1}\,, \tag{5.22}$$

where $k_1, \ldots, k_M$ are positive integers with $k_1, \ldots, k_M \geq 2$, and $(p_1, \ldots, p_N, q_1, \ldots, q_M)$ is a probability vector, $S_j x = r_j R_j x + a_j$ for $0 < r_j < 1$, R_j is an orthogonal matrix and $a_j \in \mathbb{R}^d$, and $T_j x = \rho_j P_j x + b_j$ for $0 < \rho_j < \frac{1}{\max_l k_l}$, P_j is an orthogonal matrix and $b_j \in \mathbb{R}^d$; the existence and uniqueness of measures μ satisfying (5.22) follow easily using an argument similar to the one in [GS96] or by an argument similar to the one in Proposition 2.8. As in Example 5.11 we may clearly assume that μ is not supported on any $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$.
Again, we note that measures μ satisfying the nonlinear self-similar identity in (5.22) can be viewed as inhomogeneous self-similar measures associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N, p, \nu)$, where

$$p = 1 - \sum_{j=1}^{N} p_j \quad \text{and} \quad \nu = \sum_{j=1}^{M} \frac{q_j}{1 - \sum_{k=1}^{N} p_k} (\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}}) \circ T_j^{-1}.$$

As in Example 5.11 we will apply Theorem 5.2 and Theorem 5.4 to obtain lower bounds for the infinity Fourier dimension and the 2'nd Fourier dimension of μ.

C. The infinity Fourier dimension of μ. As before we first discuss the infinity Fourier dimension of μ. Again, before analyzing this example further it is useful to make the following two observations.

Observation 5.16. *We have*

$$\underline{\Delta}_\infty(\nu) \geq \min_j k_j \underline{\Delta}_\infty(\mu). \tag{5.23}$$

Proof. First we note that for $x \in \mathbb{R}^d$

$$\begin{aligned} \widehat{\nu}(x) &= \left(\sum_{j=1}^{M} \frac{q_j}{p} (\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}}) \circ T_j^{-1} \right)^{\widehat{}} (x) \\ &= \sum_{j=1}^{M} \frac{q_j}{p} e^{i\langle b_j | x \rangle} \widehat{\mu}\left(T_j^* x\right)^{k_j}, \end{aligned} \tag{5.24}$$

where $T_j^* = \rho_j P_j^*$. It follows from (5.24) that

$$\begin{aligned} \underline{\Delta}_\infty(\nu) &= \liminf_{R \to \infty} \frac{\log\left(\sup_{|x| \geq R} \left| \sum_{j=1}^{M} \frac{q_j}{p} e^{i\langle b_j | x \rangle} \widehat{\mu}(T_j^* x)^{k_j} \right| \right)}{-\log R} \\ &\geq \liminf_{R \to \infty} \frac{\log\left(\sum_{j=1}^{M} \frac{q_j}{p} \sup_{|x| \geq R} |\widehat{\mu}(T_j^* x)|^{k_j} \right)}{-\log R}. \end{aligned} \tag{5.25}$$

Fix $\varepsilon > 0$. It now follows from the definition of $\underline{\Delta}_\infty(\mu)$ that there exists a constant $c > 0$ such that $\sup_{|x| \geq R} |\widehat{\mu}(x)| \leq c|R|^{-(\underline{\Delta}_\infty(\mu) - \varepsilon)}$ for all $R > 0$, whence $\sup_{|x| \geq R} |\widehat{\mu}(T_j^* x)|^{k_j} \leq c^{k_j} (\rho_j R)^{-k_j(\underline{\Delta}_\infty(\mu) - \varepsilon)}$ for all j. Combining this and (5.25), we conclude that

$$\underline{\Delta}_\infty(\nu) \geq \liminf_{R \to \infty} \frac{\log\left(\sum_{j=1}^{M} \frac{q_j}{p} c^{k_j} (\rho_j R)^{-k_j(\underline{\Delta}_\infty(\mu) - \varepsilon)} \right)}{-\log R}$$

$$\begin{aligned} &\geq \liminf_{R\to\infty} \frac{\log\Big(CR^{-(\min_j k_j)(\underline{\Delta}_\infty(\mu)-\varepsilon)}\Big)}{-\log R} \\ &= (\min_j k_j)(\underline{\Delta}_\infty(\mu)-\varepsilon)\,, \end{aligned}$$

where $C=\sum_{j=1}^M \frac{q_j}{p}c^{k_j}(\rho_j)^{-k_j(\underline{\Delta}_\infty(\mu)-\varepsilon)}$. Letting $\varepsilon\to 0$ we obtain (5.23). $\square$

Remark. One can clearly see that the proofs of Observation 5.16 and Observation 5.12 are very similar and therefore for the rest of this section we will omit presenting such similar proofs twice.

The next observation is due to Glickenstein and Strichartz [GS96].

Observation 5.17. *We have*

$$\underline{\Delta}_\infty(\mu)>0\,.$$

In the equicontractive case, we can use (5.23) and Corollary 5.3 to obtain $\underline{\Delta}_\infty(\mu)\geq\min(t,\underline{\Delta}_\infty(\nu))\geq\min(t,\min_j k_j\underline{\Delta}_\infty(\mu))$. Since also $\underline{\Delta}_\infty(\mu)>0$ (by Observation 5.17), we see from this inequality that $\min(t,\min_j k_j\underline{\Delta}_\infty(\mu))=t$, whence

$$\underline{\Delta}_\infty(\mu)\geq\min(t,\min_j k_j\underline{\Delta}_\infty(\mu))=t\,.$$

We now consider the general nonequicontractive case. In this case, if we proceed as in Example 5.11, we obtain the following lower bound for $\underline{\Delta}_\infty(\mu)$,

$$\underline{\Delta}_\infty(\mu) = 0 \qquad \text{if } 0\leq\underline{\Delta}_\infty(\nu)<s; \tag{5.26}$$

$$\underline{\Delta}_\infty(\mu) \geq \begin{cases}\left(1-\frac{1}{\min_j k_j}\right)s & \text{if } s\leq\underline{\Delta}_\infty(\nu)\leq t;\\ t & \text{if } t<\underline{\Delta}_\infty(\nu).\end{cases} \tag{5.27}$$

As in Example 5.11, using the fact that $\underline{\Delta}_\infty(\mu)>0$ (by Observation 5.17), we conclude that (5.26) and (5.27) simplify to: we have $s\leq\underline{\Delta}_\infty(\nu)$ and

$$\begin{aligned} \tfrac{1}{\min_j k_j}t\geq\underline{\Delta}_\infty(\mu)\geq\left(1-\tfrac{1}{\min_j k_j}\right)s &\qquad \text{if } s\leq\underline{\Delta}_\infty(\nu)\leq t,\\ \underline{\Delta}_\infty(\mu)\geq t &\qquad \text{if } t<\underline{\Delta}_\infty(\nu). \end{aligned}$$

Of course, if $1+\frac{t}{s}<\min_j k_j$, then $\frac{1}{\min_j k_j}t\geq\underline{\Delta}_\infty(\mu)\geq(1-\frac{1}{\min_j k_j})s$ cannot hold, and we must therefore have that $t<\underline{\Delta}_\infty(\nu)$. On the other hand, if $\min_j k_j\leq 1+\frac{t}{s}$, then the above result shows that $\underline{\Delta}_\infty(\mu)\geq t$ for $t<\underline{\Delta}_\infty(\nu)$, and that $\underline{\Delta}_\infty(\mu)\leq\frac{1}{\min_j k_j}t\leq\frac{1}{2}t<t$ for $s\leq\underline{\Delta}_\infty(\nu)\leq t$. However, in analogy with Glickenstein & Strichartz's result in (5.16) for $k_1=\ldots=k_M=2$, we expect that $\underline{\Delta}_\infty(\mu)\geq t$, and it therefore follows from the above that we must have $t<\underline{\Delta}_\infty(\nu)$; unfortunately, we cannot deduce this from Theorem 5.2, suggesting, once more that Theorem 5.2 can be improved.

D. The 2'nd Fourier dimension of μ. Next we analyze the 2'nd Fourier dimension of μ . Again, in order to apply Theorem 5.4 to analyze $\underline{\Delta}_2(\mu)$, we will make two further assumptions, namely that $r_1=\ldots=r_N$ and that $R_1=\ldots=R_N$. However, before analyzing the 2'nd Fourier dimension it is useful to make the following observation. The proof of this observation is similar to the proof of Observation 5.14 in Example 5.11 and is therefore omitted, see the remark following Observation 5.16.

Observation 5.18. *For all* $q > 1$, *we have*

$$\begin{aligned} \underline{\Delta}_q(\nu) &\geq \min_j k_j \underline{\Delta}_{k_j q}(\mu)\,, \\ \min_j k_j \underline{\Delta}_{k_j q}(\mu) &> \underline{\Delta}_q(\mu)\,. \end{aligned}$$

We can now use the above Observation 5.18 and Theorem 5.4 to obtain

$$\underline{\Delta}_2(\mu) \geq \min\left(t\,,\, \underline{\Delta}_2(\nu)\right) \geq \min\left(t\,,\, \min_j k_j \underline{\Delta}_{k_j 2}(\mu)\right) > \min\left(t\,,\, \underline{\Delta}_2(\mu)\right)\,.$$

Thus

$$\underline{\Delta}_2(\mu) \geq t\,.$$

This completes Example 5.15.

Example 5.19.

Finally, we consider a more concrete and rather trivial example, involving discrete measures, to support our conjectures. However, we believe that the explicit calculations for this example can be modified to consider a more interesting example involving continuous measures and therefore we have decided to include them.

For simplicity we restrict ourselves to $\mathbb{R}$. Let μ be the inhomogeneous measure satisfying the following inhomogeneous self-similar equation

$$\mu = \frac{1}{4}\mu \circ S_1^{-1} + \frac{1}{4}\mu \circ S_2^{-1} + \frac{1}{2}\delta_0\,, \tag{5.28}$$

where $S_1(x) = \frac{1}{4}(x-1)$, $S_2(x) = \frac{1}{4}(x+1)$ and δ_0 is the Dirac measure supported at 0 . It is not difficult to see that μ satisfying (5.28) is a discrete measure since in this case the measure $\nu = \delta_0$ is a discrete measure. Thus by the remark following Theorem 5.1, we have

$$\underline{\Delta}_\infty(\mu) = \underline{\Delta}_q(\mu) = 0\,. \tag{5.29}$$

We now turn towards explicit calculations and give another direct proof of (5.29). The Fourier transform $\widehat{\mu}(x)$ of μ can easily be found. Indeed, since $\widehat{\delta}_0(x) = 1$ for all x , (5.28) implies that the Fourier transform of μ satisfies the following equality

$$\widehat{\mu}(x) = \frac{1}{2}\cos\left(\frac{1}{4}x\right)\widehat{\mu}\left(\frac{1}{4}x\right) + \frac{1}{2}\,. \tag{5.30}$$

By iterating (5.30), we obtain $\widehat{\mu}(x) = \frac{1}{2^k}\prod_{l=1}^{k}\cos(\frac{1}{4^l}x)\widehat{\mu}(\frac{1}{4^k}x) + \sum_{n=1}^{k}\frac{1}{2^n}\prod_{m=1}^{n-1}\cos(\frac{1}{4^m}x)$ for all positive integers k . Letting $k \to \infty$, we obtain

$$\widehat{\mu}(x) = \sum_{n=1}^{\infty}\frac{1}{2^n}\prod_{m=1}^{n-1}\cos\left(\frac{1}{4^m}x\right)\,. \tag{5.31}$$

Figure 5.2.2 shows the graph of the Fourier transform $\widehat{\mu}(x)$ in (5.31).
Note that in this case $s = t = \frac{1}{2}$. Thus applying Corollary 5.3 gives

$$\underline{\Delta}_\infty(\mu) \geq 0\,.$$

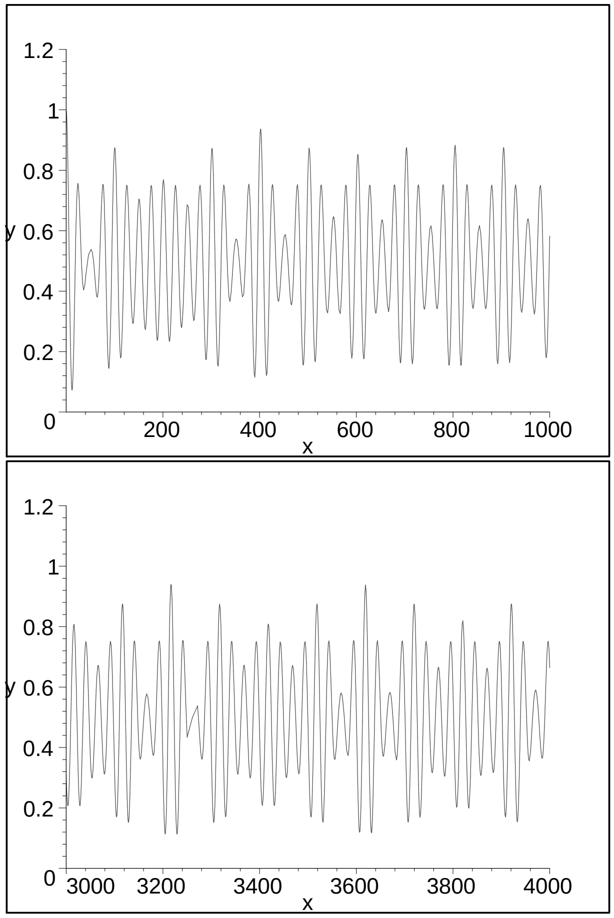

Figure 5.2.2:
This figure shows the graph of the Fourier transform $\widehat{\mu}(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}\prod_{m=1}^{n-1}\cos(\frac{1}{4^m}x)$ of the measure μ in (5.31). The top figure shows the graph of $\widehat{\mu}(x)$ for $x\in[0,1000]$ and the bottom figure shows the graph of $\widehat{\mu}(x)$ for $x\in[3000,4000]$.

We will now prove that the lower bound given by Corollary 5.3 is, in fact, the correct value of $\underline{\Delta}_\infty(\mu)$, i.e. we will prove that

$$\underline{\Delta}_\infty(\mu)=0\,. \tag{5.32}$$

For an integer n, write $x_n=8n\pi+2\pi$. Since $\cos(\frac{1}{4}x_n)=0$ for all n, we see that $\widehat{\mu}(x_n)=\frac{1}{2}+\frac{1}{2^2}\cos(\frac{1}{4}x_n)+\frac{1}{2^3}\cos(\frac{1}{4}x_n)\cos(\frac{1}{4^2}x_n)+\frac{1}{2^4}\cos(\frac{1}{4}x_n)\cos(\frac{1}{4^2}x_n)\cos(\frac{1}{4^3}x_n)+\cdots=\frac{1}{2}$ for all integers n. It follows from this that $0\le\underline{\Delta}_\infty(\mu)=\operatorname{liminf}_{R\to\infty}\frac{\log(\sup_{|x|\ge R}|\widehat{\mu}(x)|)}{-\log R}\le\operatorname{liminf}_{R\to\infty}\frac{\log(\frac{1}{2})}{-\log R}=0$, whence $\underline{\Delta}_\infty(\mu)=0$. This proves (5.32).
In fact, we will now prove that

$$\underline{\Delta}_q(\mu)=0 \tag{5.33}$$

for all $q\ge 1$. Indeed, to prove (5.33) write $I_n=[8n\pi-2\pi,8n\pi+2\pi]$ for an integer n. Since $\cos(\frac{1}{4}x)\ge 0$ for all $x\in I_n$ and all n, we see that

$$\widehat{\mu}(x) \;=\; \frac{1}{2}+\frac{1}{2^2}\cos\left(\frac{1}{4}x\right)+\frac{1}{2^3}\cos\left(\frac{1}{4}x\right)\cos\left(\frac{1}{4^2}x\right)$$

$$+\frac{1}{2^4}\cos\left(\frac{1}{4}x\right)\cos\left(\frac{1}{4^2}x\right)\cos\left(\frac{1}{4^3}x\right)+\cdots$$
$$\geq \frac{1}{2}+0-\frac{1}{2^3}-\frac{1}{2^4}-\frac{1}{2^5}-\cdots=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

for all $x \in I_n$ and all n. Writing $R_n = 8n\pi + 2\pi$, it follows from this and the fact that the intervals I_n are pairwise disjoint that

$$\int\limits_{B(0,R_n)} |\widehat{\mu}(x)|^q\,dx \geq \sum_{k=-n}^{n}\int\limits_{I_k}|\widehat{\mu}(x)|^q\,dx \geq \sum_{k=-n}^{n}\int\limits_{I_k}\frac{1}{4^q}\,dx = (2n+1)4\pi\frac{1}{4^q} \geq R_n\frac{1}{4^q}\,.$$

It follows from this that $\underline{\Delta}_q(\mu) \leq \liminf_n \frac{\log(\frac{1}{2R_n}\int_{B(0,R_n)}|\widehat{\mu}(x)|^q\,dx)^{\frac{1}{q}}}{-\log R_n} \leq \liminf_n \frac{\log(4^{-1}2^{-1/q})}{-\log R_n} = 0$. Since, clearly also $0 \leq \underline{\Delta}_q(\mu)$, we therefore conclude that $\underline{\Delta}_q(\mu) = 0$. This proves (5.33). From (5.33) we see that $\underline{\Delta}_q(\mu) = 0 = \min(\frac{1}{2}\,,0) = \min(s\,,\underline{\Delta}_q(\delta_0))$. Thus in this case the answer to Question 5.8 is affirmative. This completes Example 5.19.

5.2.4 Proof of Theorem 5.2

In this section we prove Theorem 5.2. Therefore, let the notation be as in Theorem 5.2.

Observe that it follows from (5.5) that

$$\widehat{\mu}(x) = \sum_j p_j e^{i\langle a_j|x\rangle}\widehat{\mu}(r_jR_j^*x) + p\widehat{\nu}(x)\,, \tag{5.34}$$

for $x \in \mathbb{R}^d$.
Next we introduce some notation. For $j = 1,\ldots,N$ define $P_j : \mathbb{R}^d \to \mathbb{C}$ and $L_j : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\begin{aligned} P_j(x) &= p_je^{i\langle a_j|x\rangle}\,,\\ L_j(x) &= r_jR_j^*(x)\,. \end{aligned} \tag{5.35}$$

Using this notation, (5.34) simplifies to

$$\widehat{\mu}(x) = \sum_j P_j(x)\widehat{\mu}(L_jx) + p\widehat{\nu}(x) \tag{5.36}$$

for $x \in \mathbb{R}^d$. By iterating (5.36), we see that

$$\begin{aligned} \widehat{\mu}(x) \;=\; & \sum_{j_1,\ldots,j_n=1,\ldots,N} P_{j_1}(x)\,P_{j_2}(L_{j_1}x)\cdots P_{j_n}(L_{j_{n-1}}\ldots L_{j_1}x)\,\widehat{\mu}(L_{j_n}\ldots L_{j_1}x)\\ & +p\sum_{k=0}^{n-1}\sum_{j_1,\ldots,j_k=1,\ldots,N}\Big(P_{j_1}(x)\,P_{j_2}(L_{j_1}x)\cdots\\ & \qquad\cdots P_{j_k}(L_{j_{k-1}}\ldots L_{j_1}x)\,\widehat{\nu}(L_{j_k}\ldots L_{j_1}x)\Big) \end{aligned} \tag{5.37}$$

for all positive integers n and all $x \in \mathbb{R}^d$.
Before proving Theorem 5.2 we prove auxiliary inequality (5.38) below. For a positive integer $n \geq 1$ we put

$$
\begin{aligned}
A_n &= \left\{ x \in \mathbb{R}^d \,\middle|\, |x| \geq \frac{1}{r_{\min}^n} \right\}, \\
M_n &= \sup_{x \in A_n} |\widehat{\nu}(x)| ;
\end{aligned}
$$

recall that $r_{\min} = \min_j r_j$.

Lemma 5.20. *For any* $n \geq 1$ *, we have*

$$
|\widehat{\mu}(x)| \leq (1-p)^n + \sum_{k=0}^{n-1} (1-p)^k p M_{n-k} \tag{5.38}
$$

for all $x \in A_n$

Proof. By taking absolute value in (5.37), we see that

$$
\begin{aligned}
|\widehat{\mu}(x)| \quad &\leq \sum_{j_1,\dots,j_n=1,\dots,N} |P_{j_1}(x)|\,|P_{j_2}(L_{j_1}x)| \cdots |P_{j_n}(L_{j_{n-1}} \dots L_{j_1}x)|\,|\widehat{\mu}(L_{j_n} \dots L_{j_1}x)| \\
&\qquad + p \sum_{k=0}^{n-1} \sum_{j_1,\dots,j_k=1,\dots,N} (|P_{j_1}(x)||P_{j_2}(L_{j_1}x)| \cdots \\
&\qquad\qquad \cdots |P_{j_k}(L_{j_{k-1}} \dots L_{j_1}x)|\,|\widehat{\nu}(L_{j_k} \dots L_{j_1}x)|) \\
&= \sum_{j_1,\dots,j_n=1,\dots,N} p_{j_1} p_{j_2} \cdots p_{j_n} |\widehat{\mu}(L_{j_n} \dots L_{j_1}x)| \\
&\qquad + p \sum_{k=0}^{n-1} \sum_{j_1,\dots,j_k=1,\dots,N} p_{j_1} p_{j_2} \cdots p_{j_k} |\widehat{\nu}(L_{j_k} \dots L_{j_1}x)|
\end{aligned} \tag{5.39}
$$

for $x \in \mathbb{R}^d$. Noting that $|\widehat{\mu}(x)| \leq 1$ for all x , we deduce from (5.39) that

$$
\begin{aligned}
|\widehat{\mu}(x)| \quad &\leq \quad \Big(\sum_j p_j \Big)^n + p \sum_{k=0}^{n-1} \Big(\sum_j p_j \Big)^k \sup_{y \in A_n} |\widehat{\nu}(L_{j_k} \dots L_{j_1}y)| \\
&= \quad (1-p)^n + p \sum_{k=0}^{n-1} (1-p)^k \sup_{y \in A_n} |\widehat{\nu}(L_{j_k} \dots L_{j_1}y)|
\end{aligned} \tag{5.40}
$$

for all $x \in A_n$. Finally, it is clear that if $y \in A_n$ and if $j_1, \dots, j_k = 1, \dots, N$ with $k \leq n$, then $L_{j_k} \dots L_{j_1}y \in A_{n-k}$, implying that $|\widehat{\nu}(L_{j_k} \dots L_{j_1}y)| \leq M_{n-k}$. It therefore follows from (5.40) that

$$
|\widehat{\mu}(x)| \leq (1-p)^n + \sum_{k=0}^{n-1} (1-p)^k p M_{n-k}
$$

for $x \in A_n$. This completes the proof of Lemma 5.20. $\square$

We now turn towards the proof of Theorem 5.2.

Proof of Theorem 5.2

We first show that there exists a unique number t such that

$$\sum_j p_j r_j^{-t} = 1\,, \tag{5.41}$$

and that $s = \frac{\log(1-p)}{\log r_{\min}} \le t$. Indeed, define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(x) = \sum_j p_j r_j^{-x}$. Since $\sum_j p_j < 1$ and φ is a strictly increasing continuous function with $\lim_{x\to\infty}\varphi(x) = \infty$ there exists a unique t such that (5.41) holds. Also it can be shown by routine calculations that s satisfies $\sum_j p_j r_j^{-s} < 1$. Thus $0 \le s \le t$.
For brevity we write

$$R_n = \frac{1}{r_{\min}^n}$$

throughout the remaining parts of the proof of Theorem 5.2.

Part 1: First we prove that $\underline{\Delta}_\infty(\mu) \ge \underline{\Delta}_\infty(\nu)$ *for* $0 \le \underline{\Delta}_\infty(\nu) < s$. Let $\varepsilon > 0$. Next, note that in this case $\frac{1-p}{r_{\min}^{\underline{\Delta}_\infty(\nu)-\varepsilon}} < 1$. Also, observe that it follows from the definition of $\underline{\Delta}_\infty(\nu)$ that there exists a constant $c > 0$ such that

$$\sup_{|x|\ge R} |\widehat{\nu}(x)| \le cR^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$$

for all $R > 0$. We therefore conclude from Lemma 5.20 that for all positive integers $n \ge 1$ and for all $x \in A_n$, we have

$$\begin{aligned}
|\widehat{\mu}(x)| &\le (1-p)^n + \sum_{k=0}^{n-1}(1-p)^k p M_{n-k}\\
&\le (1-p)^n + \sum_{k=0}^{n-1}(1-p)^k pc r_{\min}^{(n-k)(\underline{\Delta}_\infty(\nu)-\varepsilon)}\\
&= r_{\min}^{ns} + pc r_{\min}^{n(\underline{\Delta}_\infty(\nu)-\varepsilon)} \sum_{k=0}^{n-1}\Big(\frac{1-p}{r_{\min}^{\underline{\Delta}_\infty(\nu)-\varepsilon}}\Big)^k\\
&\le r_{\min}^{ns} + pc r_{\min}^{n(\underline{\Delta}_\infty(\nu)-\varepsilon)} \sum_{k=0}^{\infty}\Big(\frac{1-p}{r_{\min}^{\underline{\Delta}_\infty(\nu)-\varepsilon}}\Big)^k\\
&\le R_n^{-s} + CR_n^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}\\
&= (1+C)R_n^{-\min(s,\underline{\Delta}_\infty(\nu)-\varepsilon)}\\
&= (1+C)R_n^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}
\end{aligned}$$

where $C = cp/(1-\frac{1-p}{r_{\min}^{\underline{\Delta}_\infty(\nu)-\varepsilon}})$. Thus

$$\sup_{|x|\ge R_n} |\widehat{\mu}(x)| \le (1+C)R_n^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$$

for all positive integers n. Hence, if $R > 0$, we can choose a (unique) positive integer with $R_n < R \le R_{n+1}$, whence

$$\sup_{|x|\ge R} |\widehat{\mu}(x)| \le \sup_{|x|\ge R_n} |\widehat{\mu}(x)| \le (1+C)R_n^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} \le \frac{1+C}{r_{\min}^{\underline{\Delta}_\infty(\nu)-\varepsilon}} R^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$$

It follows immediately from this that $\underline{\Delta}_\infty(\mu) \ge \underline{\Delta}_\infty(\nu) - \varepsilon$. We now obtain the desired result by letting $\varepsilon \to 0$. This completes the proof of Part 1.

Part 2: Next we prove that $\underline{\Delta}_\infty(\mu) \geq \frac{s\underline{\Delta}_\infty(\nu)}{s+\underline{\Delta}_\infty(\nu)}$ *for* $s \leq \underline{\Delta}_\infty(\nu) \leq t$ *.* For positive integers n and m with $1 \leq n \leq m$ and for all $x \in A_m$, it follows from Lemma 5.20 that

$$\begin{aligned}
|\widehat{\mu}(x)| &\leq (1-p)^m + \sum_{k=0}^{n}(1-p)^k p M_{m-k} + \sum_{k=n+1}^{m}(1-p)^k p M_{m-k} \\
&\leq (1-p)^m + p\left(\frac{1}{p}\max_{m-n\leq l\leq m} M_l + \frac{(1-p)^n}{p}\sup_l M_l\right) \\
&= (1-p)^m + \sup_{|y|\geq r_{\min}^{-(m-n)}} |\widehat{\nu}(y)| + (1-p)^n ,
\end{aligned}$$

where we have used the fact that $M_l \leq 1$ for all l. Thus for any $a \geq 1$ and any $x \in A_{[an]}$ (here we write $[x]$ for the largest integer less than $x \in \mathbb{R}$), we have

$$\begin{aligned}
|\widehat{\mu}(x)| &\leq (1-p)^{[an]} + (1-p)^n + \sup_{|y|\geq r_{\min}^{-([na]-n)}} |\widehat{\nu}(y)| \\
&\leq (1-p)^{[an]} + (1-p)^n + \sup_{|y|\geq r_{\min}^{-(na-n-1)}} |\widehat{\nu}(y)| \\
&\leq 2(1-p)^n + \sup_{|y|\geq r_{\min}^{-(na-n-1)}} |\widehat{\nu}(y)| \\
&= 2r_{\min}^{ns} + \sup_{|y|\geq r_{\min}^{-n(a-1)} r_{\min}} |\widehat{\nu}(y)| \\
&\leq 2R_n^{-s} + \sup_{|y|\geq R_n^{(a-1)} r_{\min}} |\widehat{\nu}(y)| .
\end{aligned}$$

It follows from this that

$$\sup_{|x|\geq r_{\min}^{-an}} |\widehat{\mu}(x)| \leq \sup_{|x|\geq r_{\min}^{-[an]}} |\widehat{\mu}(x)| \leq 2R_n^{-s} + \sup_{|y|\geq R_n^{(a-1)} r_{\min}} |\widehat{\nu}(y)| . \tag{5.42}$$

Fix $\varepsilon > 0$. It follows from the definition of $\underline{\Delta}_\infty(\nu)$ that there exists a constant c such that

$$\sup_{|y|\geq R} |\widehat{\nu}(y)| \leq cR^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$$

for all $R > 0$. Using this and (5.42), we obtain

$$\begin{aligned}
\sup_{|x|\geq R_n^a} |\widehat{\mu}(x)| &\leq 2R_n^{-s} + c r_{\min}^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} R_n^{-(a-1)(\underline{\Delta}_\infty(\nu)-\varepsilon)} \\
&\leq \left(2 + c r_{\min}^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}\right) R_n^{-\min\left(s,(a-1)(\underline{\Delta}_\infty(\nu)-\varepsilon)\right)} \\
&= C(R_n^a)^{-\min\left(\frac{s}{a},(1-\frac{1}{a})(\underline{\Delta}_\infty(\nu)-\varepsilon)\right)}
\end{aligned}$$

where $C = 2 + c r_{\min}^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$. Hence, if $R > 0$, we can choose a (unique) positive integer with $R_n^a < R \leq R_{n+1}^a$, whence

$$\begin{aligned}
\sup_{|x|\geq R} |\widehat{\mu}(x)| &\leq \sup_{|x|\geq R_n^a} |\widehat{\mu}(x)| \leq C\,(R_n^a)^{-\min\left(\frac{s}{a},(1-\frac{1}{a})(\underline{\Delta}_\infty(\nu)-\varepsilon)\right)} \\
&\leq \frac{C}{r_{\min}^{a\min\left(\frac{s}{a},(1-\frac{1}{a})(\underline{\Delta}_\infty(\nu)-\varepsilon)\right)}} R^{-\min\left(\frac{s}{a},(1-\frac{1}{a})(\underline{\Delta}_\infty(\nu)-\varepsilon)\right)} .
\end{aligned}$$

This clearly implies that

$$\underline{\Delta}_\infty(\mu) \geq \min\left(\frac{s}{a}, \left(1-\frac{1}{a}\right)(\underline{\Delta}_\infty(\nu)-\varepsilon)\right),$$

for all $\varepsilon > 0$ and all $a \geq 1$. Letting $\varepsilon \searrow 0$ and taking supremum over all $a \geq 1$, we obtain

$$\underline{\Delta}_\infty(\mu) \geq \sup_{a\geq 1} \min\left(\frac{s}{a}, \left(1-\frac{1}{a}\right)\underline{\Delta}_\infty(\nu)\right).$$

However, it is easily seen that $\sup_{a\geq 1} \min(\frac{s}{a}, (1-\frac{1}{a})\underline{\Delta}_\infty(\nu))$ equals $\frac{s\underline{\Delta}_\infty(\nu)}{s+\underline{\Delta}_\infty(\nu)}$. Hence it follows that $\underline{\Delta}_\infty(\mu) \geq \frac{s\underline{\Delta}_\infty(\nu)}{s+\underline{\Delta}_\infty(\nu)}$. This completes the proof of Part 2.

Part 3: Finally we prove that $\underline{\Delta}_\infty(\mu) \geq t$ *for* $t < \underline{\Delta}_\infty(\nu)$. Fix $\varepsilon > 0$. It follows from the definition of t that $\sum_j p_j r_j^{-(t-\varepsilon)} < 1$. Thus, we can choose $\delta > 0$ such that

$$\sum_j p_j r_j^{-(t-\varepsilon)} + p\delta < 1. \tag{5.43}$$

Next, it follows from the definition of $\underline{\Delta}_\infty(\nu)$ that there exists a constant $c > 0$ such that

$$|\widehat{\nu}(x)| \leq c|x|^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} \tag{5.44}$$

for all x.
Also, since $\underline{\Delta}_\infty(\nu) > t$, we can clearly find $R_0 > 0$ such that

$$c|x|^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} \leq \delta|x|^{-(t-\varepsilon)} \tag{5.45}$$

for all $|x| \geq R_0$.
Finally, since $\widehat{\mu}$ is continuous, and therefore bounded on compact sets, we can find $M \geq 1$ such that $|x|^{t-\varepsilon}|\widehat{\mu}(x)| \leq M$ for all $|x| \leq R_0$, whence

$$|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)} \tag{5.46}$$

for all $|x| \leq R_0$.
Using an inductive argument we will now prove that for all integers $k \geq 0$ we have

$$|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)} \quad \text{for all } |x| \leq r_{\max}^{-k} R_0. \tag{5.47}$$

We first establish the start of the induction, namely, that (5.47) is true for $k = 0$, i.e. we prove that

$$|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)} \quad \text{for all } |x| \leq R_0. \tag{5.48}$$

However, for all $|x| \leq R_0$, it follows immediately from (5.46) that $|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)}$. This proves the start of the induction.

We now turn towards the proof of the inductive step. We therefore assume that

$$|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)} \quad \text{for all } |x| \leq r_{\max}^{-k} R_0, \tag{5.49}$$

for some fixed integer $k \geq 0$, and we must now prove that

$$|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)} \quad \text{for all } |x| \leq r_{\max}^{-(k+1)} R_0. \tag{5.50}$$

We therefore let $|x| \leq r_{\max}^{-(k+1)} R_0$ be given. It follows from the start of the induction, i.e. (5.48), that $|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)}$ for all $|x| \leq R_0$. Hence we must prove that $|\widehat{\mu}(x)| \leq M|x|^{-(t-\varepsilon)}$ for all $R_0 \leq |x| \leq r_{\max}^{-(k+1)} R_0$. Thus suppose that $R_0 \leq |x| \leq r_{\max}^{-(k+1)} R_0$. Since $|x| \leq r_{\max}^{-(k+1)} R_0$, we conclude that $|L_j x| = r_j |x| \leq r_j r_{\max}^{-(k+1)} R_0 \leq r_{\max}^{-k} R_0$ for all $j = 1, \ldots, N$, and it therefore follows from the inductive hypotheses (5.49) that

$$|\widehat{\mu}(L_j x)| \leq M |L_j x|^{-(t-\varepsilon)} = M r_j^{-(t-\varepsilon)} |x|^{-(t-\varepsilon)} . \tag{5.51}$$

Furthermore, since $R_0 \leq |x|$, it follows from (5.44) and (5.45) that

$$|\widehat{\nu}(x)| \leq c |x|^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} \leq \delta |x|^{-(t-\varepsilon)} . \tag{5.52}$$

Combining (5.51) and (5.52), we now see that

$$\begin{aligned}
|\widehat{\mu}(x)| &\leq \sum_{j=1}^{N} p_j |\widehat{\mu}(L_j x)| + p |\widehat{\nu}(x)| \\
&\leq \sum_{j=1}^{N} p_j M r_j^{-(t-\varepsilon)} |x|^{-(t-\varepsilon)} + p\delta |x|^{-(t-\varepsilon)} \\
&\leq \sum_{j=1}^{N} p_j M r_j^{-(t-\varepsilon)} |x|^{-(t-\varepsilon)} + pM\delta |x|^{-(t-\varepsilon)} \\
&= \left(\sum_{j=1}^{N} p_j r_j^{-(t-\varepsilon)} + p\delta \right) M |x|^{-(t-\varepsilon)} \\
&\leq M |x|^{-(t-\varepsilon)}
\end{aligned}$$

as required. This proves the inductive step.

Finally, (5.47) clearly implies that $\underline{\Delta}_\infty(\mu) \geq t - \varepsilon$, and letting $\varepsilon \searrow 0$ gives the desired result.

5.2.5 Proof of Theorem 5.4

In this section we prove Theorem 5.4. Therefore, let the notation be as in Theorem 5.4. Namely, again we fix a list of the form $(S_1, \ldots, S_N, p_1, \ldots, p_N, p, \nu)$ where $(p_1, \ldots, p_N, p)$ is a probability vector and $S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d$ are similarities of the form $S_j(x) = r_j R_j x + a_j$ where $0 < r_j < 1$, $a_j \in \mathbb{R}^d$ and R_j is an orthogonal matrix. We also assume that the list $(S_1, \ldots, S_N)$ satisfies the open set condition, i.e. there exists a non-empty, open and bounded set U such that $S_j(U) \subseteq U$ for all j and $S_j(U) \cap S_k(U) = \varnothing$ for all $j \neq k$. In this section will make two further assumptions, namely, we will assume that all the contracting ratios $r_1, \ldots, r_N$ are equal, and that all orthogonal matrices $R_1, \ldots, R_N$ are equal, i.e. we are assuming that

$$\begin{aligned}
r_1 &= \ldots = r_N = r , \\
R_1 &= \ldots = R_N = R .
\end{aligned}$$

We now turn towards the proof of Theorem 5.4. First we introduce some notation. Recall that s is defined by

$$s = \frac{\log(1-p)}{\log r} , \tag{5.53}$$

and define u by

$$\sum_j p_j^2 r^{-u} = 1 . \tag{5.54}$$

The proof of Theorem 5.4 will be divided into two parts. Firstly we prove that

$$\underline{\Delta}_2(\mu) \geq \frac{1}{2} \min \left(u \,,\, \sup_{\substack{q_1, q_2 > 1 \\ \frac{1}{q_1} + \frac{1}{q_2} = 1}} \Big(\min\big(\underline{\Delta}_{q_1}(\nu) \,,\, s\big) \,+\, \min\big(\underline{\Delta}_{q_2}(\nu) \,,\, s\big) \Big) \right). \tag{5.55}$$

The proof of (5.55) is given in Proposition 5.22. Secondly we prove that

$$\frac{1}{2} \min \left(u \,,\, \sup_{\substack{q_1, q_2 > 1 \\ \frac{1}{q_1} + \frac{1}{q_2} = 1}} \Big(\min\big(\underline{\Delta}_{q_1}(\nu) \,,\, s\big) \,+\, \min\big(\underline{\Delta}_{q_2}(\nu) \,,\, s\big) \Big) \right) = \min(\, s \,,\, \underline{\Delta}_2(\nu)\,). \tag{5.56}$$

The proof of (5.56) is given in Proposition 5.24. Theorem 5.4 now follows immediately by combining (5.55) and (5.56).

Below we prove (5.55). However, we begin by introducing and recalling the following notation. We let Σ^* denote the family of all finite strings $\mathbf{j} = j_1 \dots j_n$ with entries $j_k \in \{1, \dots, N\}$, i.e. $\Sigma^* = \big\{\mathbf{j} = j_1 \dots j_n \,\big|\, n \in \mathbb{N},\, j_k = 1, \dots, N\big\}$. For a finite string $\mathbf{j} = j_1 \dots j_n$ with entries $j_k \in \{1, \dots, N\}$, we will write $|\mathbf{j}|$ for the length of $\mathbf{j}$, i.e. $|\mathbf{j}| = n$ and we will write $p_{\mathbf{j}} = p_{j_1} \cdots p_{j_n}$ and $S_{\mathbf{j}} = S_{j_1} \cdots S_{j_n}$. Let $P_j : \mathbb{R}^d \to \mathbb{C}$ and $L_j : \mathbb{R}^d \to \mathbb{R}^d$ be defined as in (5.35), i.e.

$$\begin{aligned} P_j(x) &= p_j e^{i\langle a_j | x\rangle}, \\ L_j(x) &= r_j R_j^*(x) = rR^*(x). \end{aligned} \tag{5.57}$$

Finally, for $\mathbf{j} = j_1 \dots j_n \in \Sigma^*$, we write

$$a_{\mathbf{j}} = a_{j_1} + L_{j_1}^* a_{j_2} + L_{j_1}^* L_{j_2}^* a_{j_3} + \dots + L_{j_1}^* \dots L_{j_{n-1}}^* a_{j_n}. \tag{5.58}$$

Lemma 5.21. *Assume that the OSC is satisfied with open set equal to U. Assume further that $0 \in U$. Then there exists a constant $\kappa > 0$ such that*

$$|a_{\mathbf{j}_1} - a_{\mathbf{j}_2}| \geq \kappa r^n,$$

for all $\mathbf{j}_1, \mathbf{j}_2 \in \Sigma^*$ with $|\mathbf{j}_2| = |\mathbf{j}_1| = n$ and $\mathbf{j}_2 \neq \mathbf{j}_1$.

Proof. Let $\frac{\kappa}{2} = \operatorname{dist}(0, \partial U)$ and observe that $\kappa > 0$ since U is open. Since clearly $a_{\mathbf{j}} = S_{\mathbf{j}}(0)$ for all $\mathbf{j} \in \Sigma^*$, we see that $\operatorname{dist}(a_{\mathbf{j}}, \partial S_{\mathbf{j}} U) = \operatorname{dist}(S_{\mathbf{j}}(0), \partial S_{\mathbf{j}} U) = r^{|\mathbf{j}|} \operatorname{dist}(0, \partial U) = r^{|\mathbf{j}|} \frac{\kappa}{2}$. As $\mathbf{j}_1, \mathbf{j}_2 \in \Sigma^*$ with $|\mathbf{j}_2| = |\mathbf{j}_1| = n$ and $\mathbf{j}_2 \neq \mathbf{j}_1$, we conclude that $S_{\mathbf{j}_1} U$ and $S_{\mathbf{j}_2} U$ are disjoint. Thus $|a_{\mathbf{j}_1} - a_{\mathbf{j}_2}| \geq \operatorname{dist}(a_{\mathbf{j}_1}, \partial S_{\mathbf{j}_1} U) + \operatorname{dist}(a_{\mathbf{j}_2}, \partial S_{\mathbf{j}_2} U) = \frac{\kappa}{2} r^n + \frac{\kappa}{2} r^n = \kappa r^n$. This completes the proof. □

Proposition 5.22. *Assume that the OSC is satisfied. Assume that $r_1 = \ldots = r_N = r$ and that $R_1 = \ldots = R_N = R$. Recall that s and u are defined in (5.53) and (5.54), respectively. Then*

$$\underline{\Delta}_2(\mu) \geq \frac{1}{2} \min \left(u \,,\, \sup_{\substack{q_1, q_2 > 1 \\ \frac{1}{q_1} + \frac{1}{q_2} = 1}} \Big(\min\big(\underline{\Delta}_{q_1}(\nu) \,,\, s\big) \,+\, \min\big(\underline{\Delta}_{q_2}(\nu) \,,\, s\big) \Big) \right).$$

Proof. Recall that for $\mathbf{j} = j_1 \ldots j_n$, we write $a_{\mathbf{j}} = a_{j_1} + L^*_{j_1} a_{j_2} + L^*_{j_1} L^*_{j_2} a_{j_3} + \cdots + L^*_{j_1} \ldots L^*_{j_{n-1}} a_{j_n}$, and note that

$$P_{j_1}(x) P_{j_2}(L_{j_1} x) \cdots P_{j_n}(L_{j_{n-1}} \ldots L_{j_1} x) = p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} .$$

It therefore follows from (5.37) and the fact that $r_1 = \ldots = r_N = r$ and $R_1 = \ldots = R_N = R$, that

$$\begin{aligned}
\widehat{\mu}(x) &= \sum_{j_1,\ldots,j_n = 1,\ldots,N} P_{j_1}(x) P_{j_2}(L_{j_1} x) \cdots P_{j_n}(L_{j_{n-1}} \ldots L_{j_1} x) \widehat{\mu}(L_{j_n} \ldots L_{j_1} x) \\
&\qquad + p \sum_{k=0}^{n-1} \sum_{j_1,\ldots,j_k = 1,\ldots,N} \Big(P_{j_1}(x) P_{j_2}(L_{j_1} x) \cdots \\
&\qquad\qquad \cdots P_{j_k}(L_{j_{k-1}} \ldots L_{j_1} x) \widehat{\nu}(L_{j_k} \ldots L_{j_1} x) \Big) \\
&= \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \widehat{\mu}(r^n (R^n)^* x) + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \widehat{\nu}(r^k (R^k)^* x)
\end{aligned}$$

for all positive integers n and all x . Next, taking taking absolute value and recalling that $|\widehat{\mu}(x)| \le 1$ for all x , gives

$$\begin{aligned}
|\widehat{\mu}(x)| &\le |\widehat{\mu}(r^n (R^n)^* x)| \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \right| + p \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \widehat{\nu}(r^k (R^k)^* x) \right| \\
&\le \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \right| + p \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \widehat{\nu}(r^k (R^k)^* x) \right| \qquad (5.59)
\end{aligned}$$

for all positive integers n and all x . Let κ denote the constant in Lemma 5.21. It is well-known (see, e.g. [Str90b]) that we can choose two constants $c_1, c_2 > 0$ and an auxiliary function $h : \mathbb{R}^d \to \mathbb{R}$ with the following properties:

1. $h \ge 0$,
2. $h(x) \ge c_1$ for $|x| \le c_2$,
3. $\widehat{h}(0) = 1$
4. $\widehat{h}(x) = 0$ for $|x| \ge \kappa$.

Fix $q_1, q_2 > 1$ with $\frac{1}{q_1} + \frac{1}{q_2} = 1$. We now have

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int_{B(0, c_2 \frac{1}{r^n})} |\widehat{\mu}(x)|^2 \, dx \\
&\le \frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int_{B(0, c_2 \frac{1}{r^n})} \Bigg(\left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \right| \\
&\qquad + p \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \widehat{\nu}(r^k (R^*)^k x) \right| \Bigg)^2 dx \\
&\le 2 \frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int_{B(0, c_2 \frac{1}{r^n})} \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i \langle a_{\mathbf{j}} | x \rangle} \right|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\quad +2p^2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle a_{\mathbf{j}}|x\rangle} \widehat{\nu}(r^k (R^*)^k x) \right|^2 dx \\
&\leq 2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} c_1^{-1} h(r^n x) \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i\langle a_{\mathbf{j}}|x\rangle} \right|^2 dx \\
&\quad +2p^2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle a_{\mathbf{j}}|x\rangle} \widehat{\nu}(r^k (R^*)^k x) \right|^2 dx \\
&\leq 2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int c_1^{-1} h(r^n x) \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i\langle a_{\mathbf{j}}|x\rangle} \right|^2 dx \\
&\quad +2p^2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} \left| \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle a_{\mathbf{j}}|x\rangle} \widehat{\nu}(r^k (R^*)^k x) \right|^2 dx \\
&= 2c_1^{-1} \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=n} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int e^{i\langle a_{\mathbf{j}_1} - a_{\mathbf{j}_2}|x\rangle} h(r^n x)\, dx \\
&\quad +2p^2 \frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \sum_{k_1,k_2=0}^{n-1} \sum_{|\mathbf{j}_1|=k_1, |\mathbf{j}_2|=k_2} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \\
&\qquad \times \int_{B(0,c_2\frac{1}{r^n})} e^{i\langle a_{\mathbf{j}_1} - a_{\mathbf{j}_2}|x\rangle} \widehat{\nu}(r^{k_1} (R^*)^{k_1} x) \overline{\widehat{\nu}(r^{k_2} (R^*)^{k_2} x)}\, dx \\
&\leq c_0 \sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=n} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \widehat{h}(r^{-n}(a_{\mathbf{j}_1} - a_{\mathbf{j}_2})) \\
&\quad +2p^2 \sum_{k_1,k_2=0}^{n-1} \sum_{|\mathbf{j}_1|=k_1, |\mathbf{j}_2|=k_2} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \\
&\qquad \times \left(\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} |\widehat{\nu}(r^{k_1} (R^*)^{k_1} x)|^{q_1}\, dx \right)^{\frac{1}{q_1}} \\
&\qquad\qquad \times \left(\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} |\widehat{\nu}(r^{k_2} (R^*)^{k_2} x)|^{q_2}\, dx \right)^{\frac{1}{q_2}}, \quad (5.60)
\end{aligned}
$$

where $c_0 = 2c_1^{-1} \frac{1}{\mathcal{L}^d(B(0,c_2))}$ and where the last inequality is due to Hölder's inequality. Next, recall that we are assuming that the OSC is satisfied with open set equal to U, say. We may clearly assume that $0 \in U$, and it therefore follows from Lemma 5.21, that if $|\mathbf{j}_1| = |\mathbf{j}_2| = n$ with $\mathbf{j}_1 \neq \mathbf{j}_2$, then $|r^{-n}(a_{\mathbf{j}_1} - a_{\mathbf{j}_2})| \geq r^{-n}\kappa r^n = \kappa$. This and property (4) of $\widehat{h}$ therefore implies that

$$\widehat{h}(r^{-n}(a_{\mathbf{j}_1} - a_{\mathbf{j}_2})) = 0$$

for all $\mathbf{j}_1$ and $\mathbf{j}_2$ with $|\mathbf{j}_1| = |\mathbf{j}_2| = n$ and $\mathbf{j}_1 \neq \mathbf{j}_2$. Using the fact that $r^u = \sum_{j=1}^N p_j^2$, we therefore conclude that

$$\begin{aligned}\sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=n} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \widehat{h}(r^{-n}(a_{\mathbf{j}_1} - a_{\mathbf{j}_2})) &= \sum_{|\mathbf{j}|=n} p_{\mathbf{j}}^2 \\ &= \left(\sum_j p_j^2 \right)^n \\ &= r^{nu} .\end{aligned}$$

Combining this and (5.60) we now deduce that

$$\begin{aligned}&\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int\limits_{B(0, c_2 \frac{1}{r^n})} |\widehat{\mu}(x)|^2 \, dx \\ &\leq c_0 r^{nu} + 2p^2 \sum_{k_1, k_2 = 0}^{n-1} \sum_{|\mathbf{j}_1| = k_1, |\mathbf{j}_2| = k_2} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \\ &\qquad \times \left(\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int\limits_{B(0, c_2 \frac{1}{r^n})} |\widehat{\nu}(r^{k_1} (R^*)^{k_1} x)|^{q_1} \, dx \right)^{\frac{1}{q_1}} \\ &\qquad\qquad \times \left(\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int\limits_{B(0, c_2 \frac{1}{r^n})} |\widehat{\nu}(r^{k_2} (R^*)^{k_2} x)|^{q_2} \, dx \right)^{\frac{1}{q_2}} .\end{aligned}$$

Fix $\varepsilon > 0$. It follows from the definition of $\underline{\Delta}_q(\nu)$ that there exists a constant $\tilde{c} > 0$ such that

$$\left(\frac{1}{\mathcal{L}^d(B(0, \rho))} \int_{B(0, \rho)} |\widehat{\nu}(x)|^{q_j} \, dx \right)^{\frac{1}{q_j}} \leq \frac{\tilde{c}}{\rho^{\underline{\Delta}_{q_j}(\nu) - \varepsilon}}$$

for all $\rho > 0$ and $j = 1, 2$. Thus we obtain

$$\begin{aligned}&\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int\limits_{B(0, c_2 \frac{1}{r^n})} |\widehat{\mu}(x)|^2 \, dx \\ &\leq c_0 r^{nu} + 2p^2 \sum_{k_1, k_2 = 0}^{n-1} \sum_{|\mathbf{j}_1| = k_1, |\mathbf{j}_2| = k_2} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \frac{\tilde{c}}{(c_2 r^{-n+k_1})^{\underline{\Delta}_{q_1}(\nu) - \varepsilon}} \frac{\tilde{c}}{(c_2 r^{-n+k_2})^{\underline{\Delta}_{q_2}(\nu) - \varepsilon}} \\ &= c_0 r^{nu} + 2p^2 c^2 \sum_{k_1, k_2 = 0}^{n-1} \sum_{|\mathbf{j}_1| = k_1, |\mathbf{j}_2| = k_2} \frac{p_{\mathbf{j}_1}}{r^{k_1(\underline{\Delta}_{q_1}(\nu) - \varepsilon)}} \frac{p_{\mathbf{j}_2}}{r^{k_2(\underline{\Delta}_{q_2}(\nu) - \varepsilon)}} \\ &\qquad \times \left(\frac{1}{r^n} \right)^{-\underline{\Delta}_{q_1}(\nu) - \underline{\Delta}_{q_2}(\nu) + 2\varepsilon} ,\end{aligned} \tag{5.61}$$

where $c^2 = \frac{\tilde{c}^2}{c_2^{\underline{\Delta}_{q_1}(\nu) + \underline{\Delta}_{q_2}(\nu) - 2\varepsilon}}$. Momentarily writing $x_j = \frac{p_j}{r^{\underline{\Delta}_{q_1}(\nu) - \varepsilon}}$ and $y_j = \frac{p_j}{r^{\underline{\Delta}_{q_2}(\nu) - \varepsilon}}$, and $x_{\mathbf{j}} = x_{j_1} \cdots x_{j_n}$ and $y_{\mathbf{j}} = y_{j_1} \cdots y_{j_n}$ for $\mathbf{j} = j_1 \dots j_n \in \Sigma^*$, (5.61) can be written as

$$\frac{1}{\mathcal{L}^d(B(0, c_2 \frac{1}{r^n}))} \int\limits_{B(0, c_2 \frac{1}{r^n})} |\widehat{\mu}(x)|^2 \, dx$$

$$
\begin{aligned}
&\le c_0 r^{nu} \;+\; 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} \sum_{k_1,k_2=0}^{n-1} \sum_{|\mathbf{j}_1|=k_1,|\mathbf{j}_2|=k_2} x_{\mathbf{j}_1} y_{\mathbf{j}_2} \\
&= c_0 r^{nu} \;+\; 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} \left(\sum_{k=0}^{n-1}\sum_{|\mathbf{j}|=k} x_{\mathbf{j}}\right)\left(\sum_{k=0}^{n-1}\sum_{|\mathbf{j}|=k} y_{\mathbf{j}}\right) \\
&= c_0 r^{nu} + 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} \left(\sum_{k=0}^{n-1}\left(\sum_{j} x_j\right)^k\right)\left(\sum_{k=0}^{n-1}\left(\sum_{j} y_j\right)^k\right) \\
&= c_0 r^{nu} \\
&\quad + 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} \left(\sum_{k=0}^{n-1}\left(\frac{1-p}{r^{\underline{\Delta}_{q_1}(\nu)-\varepsilon}}\right)^k\right)\left(\sum_{k=0}^{n-1}\left(\frac{1-p}{r^{\underline{\Delta}_{q_2}(\nu)-\varepsilon}}\right)^k\right).
\end{aligned} \tag{5.62}
$$

It is easily seen that if $\underline{\Delta}_{q_j}(\nu)-\varepsilon \le s$, then there exists an N_ε such that if $n \ge N_\varepsilon$, then

$$
\sum_{k=0}^{n-1}\left(\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}\right)^k = \begin{cases} \dfrac{1-\left(\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}\right)^n}{1-\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}} \le \dfrac{1}{1-\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}} \le \frac{1}{r^{n\varepsilon}} & \text{for } \underline{\Delta}_{q_j}(\nu)-\varepsilon < s; \\ & \\ n \le \frac{1}{r^{n\varepsilon}} & \text{for } \underline{\Delta}_{q_j}(\nu)-\varepsilon = s, \end{cases} \tag{5.63}
$$

for $j=1,2$. Also, if $s < \underline{\Delta}_{q_j}(\nu)-\varepsilon$, then

$$
\sum_{k=0}^{n-1}\left(\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}\right)^k = \frac{1-\left(\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}\right)^n}{1-\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}} \le \frac{\left(\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}\right)^n}{\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}-1} \le C_j \frac{1}{r^{(\underline{\Delta}_{q_j}(\nu)-\varepsilon-s)n}} \tag{5.64}
$$

where $C_j = \frac{1}{\frac{1-p}{r^{\underline{\Delta}_{q_j}(\nu)-\varepsilon}}-1}$. Combining (5.62), (5.63) and (5.64) and putting $C = c_0 + 2p^2c^2\max(1, C_1, C_2, C_1C_2)$ give the following:

If $\underline{\Delta}_{q_1}(\nu)-\varepsilon \le s$ and $\underline{\Delta}_{q_2}(\nu)-\varepsilon \le s$, then

$$
\begin{aligned}
\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))}\int_{B(0,c_2\frac{1}{r^n})} |\widehat{\mu}(x)|^2\,dx \;&\le\; c_0 r^{nu} \;+\; 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon}\frac{1}{r^{2n\varepsilon}} \\
&\le\; C\left(\frac{1}{r^n}\right)^{-\min\left(u\,,\,\underline{\Delta}_{q_1}(\nu)+\underline{\Delta}_{q_2}(\nu)-4\varepsilon\right)}
\end{aligned} \tag{5.65}
$$

for all $n \ge N_\varepsilon$;

If $s < \underline{\Delta}_{q_1}(\nu)-\varepsilon$ and $s < \underline{\Delta}_{q_2}(\nu)-\varepsilon$, then

$$
\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))}\int_{B(0,c_2\frac{1}{r^n})} |\widehat{\mu}(x)|^2\,dx \\
&\quad\le c_0 r^{nu} \;+\; 2p^2c^2\left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} C_1C_2\frac{1}{r^{(\underline{\Delta}_{q_1}(\nu)-\varepsilon-s)n}}\frac{1}{r^{(\underline{\Delta}_{q_2}(\nu)-\varepsilon-s)n}}
\end{aligned}
$$

$$\leq C \left(\frac{1}{r^n}\right)^{-\min(u\,,2s)} \quad ; \tag{5.66}$$

If $\underline{\Delta}_{q_2}(\nu) - \varepsilon \leq s < \underline{\Delta}_{q_1}(\nu) - \varepsilon$, then

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} |\widehat{\mu}(x)|^2\,dx \\
&\leq c_0 r^{nu} \ + \ 2p^2c^2 \left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} C_1 \frac{1}{r^{(\underline{\Delta}_{q_1}(\nu)-\varepsilon-s)n}}\frac{1}{r^{n\varepsilon}} \\
&\leq C \left(\frac{1}{r^n}\right)^{-\min\left(u\,,\underline{\Delta}_{q_2}(\nu)+s-2\varepsilon\right)}
\end{aligned} \tag{5.67}$$

for all $n \geq N_\varepsilon$;

If $\underline{\Delta}_{q_1}(\nu) - \varepsilon \leq s < \underline{\Delta}_{q_2}(\nu) - \varepsilon$, then

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,c_2\frac{1}{r^n}))} \int_{B(0,c_2\frac{1}{r^n})} |\widehat{\mu}(x)|^2\,dx \\
&\leq c_0 r^{nu} \ + \ 2p^2c^2 \left(\frac{1}{r^n}\right)^{-\underline{\Delta}_{q_1}(\nu)-\underline{\Delta}_{q_2}(\nu)+2\varepsilon} C_2 \frac{1}{r^{(\underline{\Delta}_{q_2}(\nu)-\varepsilon-s)n}}\frac{1}{r^{n\varepsilon}} \\
&\leq C \left(\frac{1}{r^n}\right)^{-\min\left(u\,,\underline{\Delta}_{q_1}(\nu)+s-2\varepsilon\right)}
\end{aligned} \tag{5.68}$$

for all $n \geq N_\varepsilon$.

The desired result follows easily from (5.65)–(5.68). □

Lemma 5.23. *Let* λ *be a Borel probability measure on* $\mathbb{R}^d$. *Let* $q_1, q_2 > 1$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = 1$. *Then*

$$\underline{\Delta}_{q_1}(\lambda) + \underline{\Delta}_{q_2}(\lambda) \leq 2\underline{\Delta}_2(\lambda)\,.$$

Proof. For real numbers q and ρ with $q, \rho > 1$, and a bounded measurable function $f : \mathbb{R}^d \to \mathbb{C}$, let $\|f\|_{q,\rho} = (\frac{1}{\mathcal{L}^d(B(0,\rho))}\int_{B(0,\rho)} |f(x)|^q\,dx)^{\frac{1}{q}}$ denote the q'th norm of f with respect to normalized Lebesgue measure restricted to the ball $B(0,\rho)$; recall, that $\mathcal{L}^d$ denotes Lebesgue measure in $\mathbb{R}^d$. With this notation we see that $\underline{\Delta}_q(\lambda) = \liminf_{\rho\to\infty} \frac{\log\|\widehat{\lambda}\|_{q,\rho}}{-\log\rho}$ for all $q \geq 1$. It now follows from Hölder's inequality that $\|\widehat{\lambda}^2\|_{1,\rho} \leq \|\widehat{\lambda}\|_{q_1,\rho}\|\widehat{\lambda}\|_{q_2,\rho}$, whence

$$\begin{aligned}
\underline{\Delta}_{q_1}(\lambda) + \underline{\Delta}_{q_2}(\lambda) &= \liminf_{\rho\to\infty}\frac{\log\|\widehat{\lambda}\|_{q_1,\rho}}{-\log\rho} + \liminf_{\rho\to\infty}\frac{\log\|\widehat{\lambda}\|_{q_2,\rho}}{-\log\rho} \\
&\leq \liminf_{\rho\to\infty}\left(\frac{\log\|\widehat{\lambda}\|_{q_1,\rho}}{-\log\rho} + \frac{\log\|\widehat{\lambda}\|_{q_2,\rho}}{-\log\rho}\right) \\
&= \liminf_{\rho\to\infty}\frac{\log\left(\|\widehat{\lambda}\|_{q_1,\rho}\,\|\widehat{\lambda}\|_{q_2,\rho}\right)}{-\log\rho} \\
&\leq \liminf_{\rho\to\infty}\frac{\log\|\widehat{\lambda}^2\|_{1,\rho}}{-\log\rho} = \liminf_{\rho\to\infty}\frac{\log\|\widehat{\lambda}\|_{2,\rho}^2}{-\log\rho} = 2\underline{\Delta}_2(\lambda)\,.
\end{aligned}$$

This proves Lemma 5.23. □

Proposition 5.24. *Let λ be a probability measure on $\mathbb{R}^d$. Recall that s and u are defined in (5.53) and (5.54), respectively. Then*

$$\frac{1}{2}\min\left(u\ ,\ \sup_{\substack{q_1,q_2>1\\ \frac{1}{q_1}+\frac{1}{q_2}=1}} \Big(\ \min\big(\underline{\Delta}_{q_1}(\lambda)\,,\,s\big)\ +\ \min\big(\underline{\Delta}_{q_2}(\lambda)\,,\,s\big)\ \Big)\right) = \min\big(\,s\,,\,\underline{\Delta}_2(\lambda)\big)\ .$$

Proof. For brevity write

$$\Delta = \frac{1}{2}\min\left(u\ ,\ \sup_{\substack{q_1,q_2>1\\ \frac{1}{q_1}+\frac{1}{q_2}=1}} \Big(\ \min\big(\underline{\Delta}_{q_1}(\lambda)\,,\,s\big)\ +\ \min\big(\underline{\Delta}_{q_2}(\lambda)\,,\,s\big)\ \Big)\right).$$

Next observe that $r^u = \sum_j p_j^2 \le (\sum_j p_j)^2 = (1-p)^2$, whence $u \ge 2\frac{\log(1-p)}{\log r} = 2s$.
Part 1: We prove that $\Delta \ge \min(s, \underline{\Delta}_2(\lambda))$. Putting $q_1 = q_2 = 2$ in the supremum in Δ and using the fact that $u > 2s$ gives

$$\begin{aligned}
\Delta &\ \ge\ \frac{1}{2}\min\big(\,u\ ,\ \min\big(\underline{\Delta}_2(\lambda)\,,\,s\big)\ +\ \min\big(\underline{\Delta}_2(\lambda)\,,\,s\big)\big)\\
&\ =\ \frac{1}{2}\min\big(\,u\ ,\ 2\min\big(\underline{\Delta}_2(\lambda)\,,\,s\big)\big)\\
&\ =\ \min\big(\,s\,,\,\underline{\Delta}_2(\lambda)\big)\ .
\end{aligned}$$

Part 2: We prove that $\Delta \le \min(s, \underline{\Delta}_2(\lambda))$. Using Lemma 5.23 and the fact that $u > 2s$ gives

$$\begin{aligned}
\Delta &\ \le\ \frac{1}{2}\min\left(u\ ,\ \sup_{\substack{q_1,q_2>1\\ \frac{1}{q_1}+\frac{1}{q_2}=1}} \Big(\ \min\big(\underline{\Delta}_{q_1}(\lambda)+\underline{\Delta}_{q_2}(\lambda)\,,\,2s\big)\ \Big)\right)\\
&\ \le\ \frac{1}{2}\min\left(u\ ,\ \sup_{\substack{q_1,q_2>1\\ \frac{1}{q_1}+\frac{1}{q_2}=1}} \Big(\ \min\big(2\underline{\Delta}_2(\lambda)\,,\,2s\big)\ \Big)\right)\\
&\ =\ \min\big(\,s\,,\,\underline{\Delta}_2(\lambda)\big)\ .
\end{aligned}$$

This completes the proof of Proposition 5.24. $\square$

Proof of Theorem 5.4
The proof of Theorem 5.4 follows immediately from Proposition 5.22 and Proposition 5.24.

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